CHAPTER 7

Utility Theory

7.1. Single period utility theory

We wish to use a concept of utility that is able to deal with uncertainty. So we introduce the von Neumann–Morgenstern utility function. The investor makes choices consistent with maximizing the expected value of the utility function. Utility is defined over either consumption or wealth, but can also be defined over a bundle of goods. von Neumann-Morgenstern utility functions can be developed axiomatically.

Suppose an individual cares about consumption c and define a lottery L as a set of consumption outcomes $(c_1, \ldots, c_m)$ and an associated set of probabilities $(\pi_1, \ldots, \pi_m)$. Let $\mathcal{L}$ be the space of lotteries. Elements of the set of consumption outcomes can be elements of $\mathcal{L}$.

Axiom 1 (Completeness): For every pair of lotteries, $L_1 \succsim L_2$ or $L_1 \precsim L_2$.

Axiom 2 (Reflexivity): For every lottery, $L \succsim L$.

Axiom 3 (Transitivity): If $L_1 \succsim L_2$ and $L_2 \succsim L_3$, then $L_1 \succsim L_3$.

Axiom 4 (Continuity): For any $x, y, z \in \mathcal{L}$,

$$\{\pi \in [0,1] : ((x, y), (\pi, (1-\pi))) \succsim (z, 1)\}$$

and

$$\{\pi \in [0,1] : ((x, y), (\pi, (1-\pi))) \precsim (z, 1)\}$$

are closed sets.

Axiom 5 (Independence): If $(x, 1) \sim (y, 1)$, then $((x, z), (\pi, 1-\pi)) \sim ((y, z), (\pi, 1-\pi))$.

Axiom 6 (Dominance): Let $L_1 = ((x_1, x_2), (\pi_1, 1-\pi_1))$ and $L_2 = ((x_1, x_2), (\pi_2, 1-\pi_2))$. If $x_1 \succ x_2$, then $L_1 \succ L_2 \Leftrightarrow \pi_1 > \pi_2$.

Theorem 27. *Under Axioms 1–6, there exists a utility function $u(\cdot)$ such that the choice of lottery by a decisionmaker corresponds to the lottery with the highest* $\mathrm{E}[u(\cdot)]$.

PROOF. Let L_h be the best lottery in $\mathcal{L}$ and L_w be the worst lottery in $\mathcal{L}$. Define $u(L_h) = 1$ and $u(L_w) = 0$. To find the utility of an intermediate lottery $L_i \in \mathcal{L}$, set $u(L_i) = p_i$, where p_i is defined by $((L_h, L_w), (p_i, 1-p_i)) \sim (L_i, 1)$.

We need to check two things:

- Existence of such a p_i: The two sets

$$\{p \in [0,1] : ((L_h, L_w), (p, 1-p)) \succsim L_i\}$$

and

$$\{p \in [0,1] : ((L_h, L_w), (p, 1-p)) \precsim L_i\}$$

are closed (Axiom 4) and nonempty (since L_h is best and L_w is worst). Every point in [0, 1] is in one or the other of the two sets (Axiom 1). Since the unit interval is connected, there must be some p in both, but this will be just the desired p_i.

- Uniqueness of such a p_i: Suppose p_i and p_i' both satisfy the above definition. Then one must be larger than the other. By Axiom 6, the lottery that gives a bigger probability of L_h must be preferred to the one that gives a smaller probability. Hence p_i is unique and $u(\cdot)$ is well defined.

We next check that $u(\cdot)$ has the expected utility property. This can be shown as follows. By Axiom 5,

$$((L_x, L_y), (p, 1-p)) \sim ((((L_h, L_w), (p_x, 1-p_x)), ((L_h, L_w), (p_y, 1-p_y))), (p, 1-p)),$$

$$\sim ((L_h, L_w), (p_x p + p_y(1-p), \underbrace{(1-p_x)p + (1-p_y)(1-p)}_{1-p_x p - p_y(1-p)}))$$

$$\sim ((L_h, L_w), (pu(L_x) + (1-p)u(L_y), 1 - pu(L_x) - (1-p)u(L_y))),$$

so

$$u((L_x, L_y), (p, 1-p)) = pu(L_x) + (1-p)u(L_y),$$

as required.

Finally we verify that $u(\cdot)$ is a utility function. Suppose that $L_x \succ L_y$. Then

$$u(L_x) = p_x \ni L_x \sim ((L_h, L_w), (p_x, 1-p_x)),$$
$$u(L_y) = p_y \ni L_y \sim ((L_h, L_w), (p_y, 1-p_y)),$$

so by Axiom 6, $u(L_x) > u(L_y)$. □

A von Neumann–Morgenstern (vNM) utility function is identified up to a positive linear transformation, i.e. $u(\cdot)$ and $a + bu(\cdot)$ with $b > 0$ are equivalent. If the agent prefers more to less, $u(\cdot)$ is increasing.

7.2. Risk aversion

A decision-maker is said to be risk averse at a particular consumption level if she is *not* prepared to accept any actuarially fair, immediately resolved consumption gamble. For a given vNM utility function $u(c\cdot)$, the utility function is risk averse at c if $u(c) > \mathrm{E}[u(c+\varepsilon)]$ for all ε satisfying $\mathrm{E}[\varepsilon] = 0$ and $\mathrm{var}[\varepsilon] > 0$.

Theorem 28. *A decisionmaker is globally risk averse iff her vNM utility function is strictly concave at all consumption levels.*

Absolute risk aversion. For a utility function $u(\cdot)$, the absolute risk aversion measure is defined as

$$ARA(c) = -\frac{u''(c)}{u'(c)}.$$

It measures the dollar amount that the decision-maker would pay to avoid an actually fair gamble (per unit of dollar gamble variance), i.e.

$$\mathrm{E}[u(c+\varepsilon)] \approx u(c - \underbrace{\tfrac{1}{2}ARA(c)\,\mathrm{var}[\varepsilon]}_{q}),$$

where $\mathrm{E}[\varepsilon] = 0$. To show this, use Taylor series expansions of both sides:

$$\begin{aligned}\mathrm{E}[u(c) + \varepsilon u'(c) + \tfrac{1}{2}\varepsilon^2 u''(c)] &\approx u(c) - qu'(c)\\ \Rightarrow \tfrac{1}{2}\,\mathrm{var}[\varepsilon]u''(c) &\approx -qu'(c),\\ q &= \tfrac{1}{2}ARA(c)\,\mathrm{var}[\varepsilon].\end{aligned}$$

Relative risk aversion. For a utility function $u(\cdot)$, the relative risk aversion measure is defined as

$$RRA(c) = -\frac{u''(c)}{u'(c)}c.$$

It measures the amount (as a fraction of consumption) that the decision-maker would pay to avoid an actuarially fair gamble (per unit of variance of the gamble expressed as a fraction of consumption), i.e.

$$\mathrm{E}[u(c+e)] \approx u(c - \underbrace{c\tfrac{1}{2}RRA(c)\,\mathrm{var}[\varepsilon/c]}_{q})),$$

where $\mathrm{E}[\varepsilon] = 0$. To show this, notice that

$$\tfrac{1}{2}cRRA(c)\,\mathrm{var}[\varepsilon/c] = \tfrac{1}{2}ARA(c)\,\mathrm{var}[\varepsilon].$$

Comparison of $ARA(c)$ and $RRA(c)$. Consider:

- *CARA*: constant $ARA(c)$, so $RRA(c)$ is increasing in c.
- *CRRA*: constant $RRA(c)$, so $ARA(c)$ is decreasing in c.

Fix ε, vary c, and consider the dollar payment the decision-maker would pay:

$$\begin{aligned}q_{CARA(c)} &= \tfrac{1}{2}CARA\,\mathrm{var}[\varepsilon] \quad \text{a constant,}\\ q_{CRRA(c)} &= \tfrac{1}{2}\frac{CRRA}{c}\,\mathrm{var}[\varepsilon] \quad \text{which is decreasing in } c.\end{aligned}$$

Fix ε/c, vary c, and consider the payment as a fraction of c that the decision-maker would pay:

$$\begin{aligned}q_{CARA}(c)/c &= \tfrac{1}{2}cCARA\,\mathrm{var}[\varepsilon/c] \quad \text{which is increasing in } c,\\ q_{CRRA}(c)/c &= \tfrac{1}{2}CRRA\,\mathrm{var}[\varepsilon/c] \quad \text{a constant.}\end{aligned}$$

7.3. HARA utility function

The functional form of the hyperbolic absolute risk aversion (HARA) utility function is

$$u(w) = a\left(\frac{1-\varphi}{\varphi}\right)\left(\frac{w}{1-\varphi} - \hat{w}\right)^{\varphi} + b, \qquad \varphi \neq 0, \quad \frac{w}{1-\varphi} - \hat{w} > 0, \quad a > 0,$$

with

$$\begin{aligned} u'(w) &= a\left(\frac{w}{1-\varphi} - \hat{w}\right)^{\varphi-1} > 0, \\ u''(w) &= -a\left(\frac{w}{1-\varphi} - \hat{w}\right)^{\varphi-2} < 0, \\ ARA(w) &= -\frac{u''(w)}{u'(w)} = \left(\frac{w}{1-\varphi} - \hat{w}\right)^{-1}, \\ RRA(w) &= -\frac{u''(w)w}{u'(w)} = w\left(\frac{w}{1-\varphi} - \hat{w}\right)^{-1}. \end{aligned}$$

HARA specializes to a number of important utility specifications.

Power utility. If $\hat{w} = 0$, we obtain power utility. Defining $\gamma = 1-\varphi$ and setting $a = (1-\varphi)^{-1+\varphi}$, $1-\varphi > 0$, then

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0, \quad w > 0, \quad RRA = \gamma.$$

Log utility. If $\hat{w} = 0$, $a = (1-\varphi)^{-1+\varphi}$ and $b = -1/\varphi$, letting $\varphi \to 0$, we obtain log utility as the limit since

$$u(w) = \frac{w^{\varphi} - 1}{\varphi},$$

so using L'Hôpital's rule,

$$\lim_{\varphi \to 0} u(w) = \lim_{\varphi \to 0} \frac{\frac{d}{d\varphi}(e^{\varphi \ln w} - 1)}{\frac{d}{d\varphi}\varphi} = \lim_{\varphi \to 0} \frac{w^{\varphi} \ln w}{1} = \ln w.$$

Note that since $u'(w) = \infty$, a power or log utility agent does not want to hold a wealth portfolio with a positive probability of a return of -1. Consequently, when a power or log utility agent has access to risky assets whose returns are multivariate normal and a riskless asset, the agent chooses to hold a portfolio 100% invested in the riskless asset.

Quadratic utility. If $\varphi = 2$ and $\hat{w} < 0$, we obtain quadratic utility. Setting $a = 1$ and $b = 0$ and defining $w^* = -\hat{w}$, we get

$$u(w) = -\tfrac{1}{2}(w^* - w)^2, \quad w < w^*.$$

Exponential utility. If

$$\hat{w} = \left(\frac{\varphi}{1-\varphi}\right)\frac{1}{\lambda}, \quad a = \left(\frac{\varphi-1}{\varphi}\right)^{\varphi-1}\lambda^{\varphi}, \quad b = 0,$$

then letting $\varphi \to \infty$, we obtain exponential utility since

$$\begin{aligned} u(w) &= \left(\frac{\varphi-1}{\varphi}\right)^{\varphi-1}\lambda^{\varphi}\frac{(1-\varphi)}{\varphi}\left(\frac{w}{1-\varphi} - \left(\frac{\varphi}{1-\varphi}\right)\frac{1}{\lambda}\right)^{\varphi} \\ &= -\left(\frac{\varphi-1}{\varphi}\right)^{\varphi}\lambda^{\varphi}\left(\frac{1}{\varphi-1}\left(\frac{\varphi}{\lambda} - w\right)\right)^{\varphi} \\ &= -\left(1 - \frac{\lambda w}{\varphi}\right)^{\varphi}, \end{aligned}$$

and so

$$\lim_{\varphi\to\infty} u(w) = \lim_{\varphi\to\infty} -\left(1 - \frac{\lambda w}{\varphi}\right)^{\varphi} = -e^{-\lambda w}, \quad ARA(w) = \lambda.$$

7.4. Utility theory with multiple periods

Use a time separable (or additive) utility function

$$\max \mathrm{E}\left[\sum_{\tau=0}^{\infty}\delta^{\tau}u(c_{t+\tau})\,\middle|\,\mathcal{F}_t\right],$$

where $u(\cdot)$ is a vNM utility function and δ is a patience parameter.

Intertemporal elasticity of substitution. The intertemporal elasticity of substitution is the percentage change in consumption growth in response to a percentage change in one plus the interest rate.

With power utility, the intertemporal elasticity of substitution for consumption is equal to the inverse of RRA. Here are two illustrations.

- Ignore uncertainty and consider a 2-period problem:

  $$\max_{c_t, c_{t+1}} \frac{c_t^{1-\gamma}}{1-\gamma} + \delta\frac{c_{t+1}^{1-\gamma}}{1-\gamma} \quad \text{s.t. } c_{t+1} = (W_t - c_t)(1+R).$$

  The first-order condition is

  $$\begin{aligned} & c_t^{-\gamma} - \delta(1+R)c_{t+1}^{-\gamma} = 0 \\ \Rightarrow & \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \delta(1+R) \\ \Rightarrow & \frac{c_{t+1}}{c_t} = \delta^{1/\gamma}(1+R)^{1/\gamma} \\ \Rightarrow & \ln\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\gamma}\ln\delta + \frac{1}{\gamma}\ln(1+R). \end{aligned}$$

  Now the elasticity is given by

  $$\frac{d\ln\frac{c_{t+1}}{c_t}}{d\ln(1+R)} = \frac{1}{\gamma},$$

as required. Why?

$$\frac{d\,\ln(\frac{c_{t+1}}{c_t})}{d\,\ln(1+R)} = \frac{\frac{d\,\ln\frac{c_{t+1}}{c_t}}{d\,\frac{c_{t+1}}{c_t}}}{\frac{d\,\ln(1+R)}{d(1+R)}}\frac{d\,\frac{c_{t+1}}{c_t}}{d(1+R)} = \frac{d\,\frac{c_{t+1}}{c_t}\Big/\frac{c_{t+1}}{c_t}}{d(1+R)/(1+R)}.$$

- Consider a 2-period problem and assume consumption growth is log normal conditional on $\mathcal{F}_t$:

$$\max_{c_t, c_{t+1}} \frac{c_t^{1-\gamma}}{1-\gamma} + \delta\,\mathrm{E}_t[\frac{c_{t+1}^{1-\gamma}}{1-\gamma}] \quad \text{s.t. } c_{t+1} = (W_t - c_t)(1+R).$$

The first-order condition is

$$c_t^{-\gamma} - \delta(1+R)\,\mathrm{E}_t[c_{t+1}^{-\gamma}] = 0$$
$$\Rightarrow \delta\,\mathrm{E}_t[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}(1+R)] = 1$$
$$\Rightarrow \mathrm{E}_t[e^{\ln(\delta)-\gamma\ln(c_{t+1}/c_t)+\ln(1+R)}] = 1$$
$$\Rightarrow e^{\ln(\delta-\gamma\,\mathrm{E}_t[\ln(c_{t+1}/c_t)]+\ln(1+R)+0.5\sigma_t^2[\ln(c_{t+1}/c_t)]} = 1$$
$$\Rightarrow \ln(\delta) - \gamma\;E_t[\ln(c_{t+1}/c_t)] + \ln(1+R) + 0.5\sigma_t^2[\ln(c_{t+1}/c_t)] = 0$$
$$\Rightarrow \mathrm{E}_t[\ln(c_{t+1}/c_t)] = \frac{1}{\gamma}\ln(\delta) + \frac{1}{\gamma}\ln(1+R) + \frac{0.5}{\gamma}\sigma_t^2[\ln(c_{t+1}/c_t)] = 0.$$

So the elasticity is given by

$$\frac{dE_t[\ln(\frac{c_{t+1}}{c_t})]}{d\,\ln(1+R)} = \frac{1}{\gamma},$$

as required.

7.5. Using dynamic programming to solve multiperiod problems

Finite-lived investor. Define

$$V_t(W_t, s_t) = \max_{\{c_\tau\}_{\tau=t}^{T-1}, \{\boldsymbol{\alpha}_\tau\}_{\tau=t}^{T-1}} \mathrm{E}\left[\sum_{\tau=t}^{T}\delta^{\tau-t}u(c_\tau)\,|\,s_t, W_t\right] \tag{27}$$

subject to

$$W_{\tau+1} = (W_\tau - c_\tau)(\boldsymbol{\alpha}'_\tau(\boldsymbol{R}_{\tau+1} - \mathbf{i}_N R^f_{\tau+1}) + R^f_{\tau+1}), \quad c_\tau, \boldsymbol{\alpha}_\tau \in \mathcal{F}_\tau(s_\tau, W_\tau) \tag{28}$$

for $\tau = t, \ldots, T-1$, where s_τ is the vector of state variables that affect the investor's happiness at time τ for $\tau = t, \ldots, T-1$.

We would like to convert the multiperiod problem into a single period problem. We can solve recursively working back from T. At time $(T-1)$,

$$V_{T-1}(W_{T-1}, s_{T-1}) = \max_{c_{T-1}, \boldsymbol{\alpha}_{T-1}} \mathrm{E}[u(c_{T-1}) + \delta u(W_T)\,|\,s_{T-1}, W_{T-1}]$$

subject to

$$W_T = (W_{T-1} - c_{T-1})(\boldsymbol{\alpha}'_{T-1}(\boldsymbol{R}_T - \mathbf{i}_N R^f_T) + R^f_T), \quad c_{T-1}, \boldsymbol{\alpha}_{T-1} \in \mathcal{F}(s_{T-1}).$$

At $T-2$,

$$V_{T-2}(W_{T-2}, s_{T-2}) = \max_{\substack{\{c_\tau\}_{\tau=T-2}^{T-1},\\ \{\boldsymbol{\alpha}\}_{\tau=T-2}^{T-1}}} \left\{ \mathrm{E}\left[u(c_{T-2}) + \delta u(c_{T-1}) + \delta^2 u(W_T) \,\middle|\, \begin{matrix} s_{T-2} \\ W_{T-2} \end{matrix} \right] \right\}$$

subject to

$$W_{\tau+1} = (W_\tau - c_\tau)(\boldsymbol{\alpha}'_\tau(\boldsymbol{R}_{\tau+1} - \mathbf{i}_N R^f_{\tau+1}) + R^f_{\tau+1}), \quad c_\tau, \boldsymbol{\alpha}_\tau \in \mathcal{F}(s_\tau)$$

for $\tau = T-2, T-1$. This time-($T-2$) value function can be written as:

$$V_{T-2}(W_{T-2}, s_{T-2}) =$$

$$\max_{\substack{c_{T-2},\\ \boldsymbol{\alpha}_{T-2}}} \left\{ \mathrm{E}\left[u(c_{T-2}) + \delta \underbrace{\max_{c_{T-1}, \boldsymbol{\alpha}_{T-1}} \left\{ \begin{matrix} \mathrm{E}\left[u(c_{T-1}) + \delta u(W_T) \,\middle|\, \begin{matrix} s_{T-1} \\ W_{T-1} \end{matrix} \right] \\ \text{s.t. } W_T = (W_{T-1} - c_{T-1}) \times \\ (\boldsymbol{\alpha}'_{T-1}(\boldsymbol{R}_T - \mathbf{i}_N R^f_T) + R^f_T), \\ c_{T-1}, \boldsymbol{\alpha}_{T-1} \in \mathcal{F}(s_{T-1}) \end{matrix} \right\}}_{V_{T-1}(W_{T-1}, s_{T-1})} \,\middle|\, \begin{matrix} s_{T-2} \\ W_{T-2} \end{matrix} \right] \right\}$$

subject to

$$W_{T-1} = (W_{T-2} - c_{T-2})(\boldsymbol{\alpha}'_{T-2}(\boldsymbol{R}_{T-1} - \mathbf{i}_N R^f_{T-1}) + R^f_{t-1}), \quad c_{T-2}, \boldsymbol{\alpha}_{T-2} \in \mathcal{F}(s_{T-2}).$$

Thus, the time-($T-2$) value function becomes

$$V_{T-2}(W_{T-2}, s_{T-2}) = \max_{c_{T-2}, \boldsymbol{\alpha}_{T-2}} \left\{ \mathrm{E}\left[u(c_{T-2} + \delta V_{T-1}(W_{T-1}, s_{T-1}) \,\middle|\, \begin{matrix} s_{T-2} \\ W_{T-2} \end{matrix} \right] \right\}$$

subject to

$$W_{T-1} = (W_{T-2} - c_{T-2})(\boldsymbol{\alpha}'_{T-2}(\boldsymbol{R}_{T-1} - \mathbf{i}_N R^f_{T-1}) + R^f_{T-1}), \quad c_{T-2}, \boldsymbol{\alpha}_{T-2} \in \mathcal{F}(s_{T-2}).$$

This holds for any $\tau = t, \ldots, T-1$, so

$$V_\tau(W_\tau, s_\tau) = \max_{c_\tau, \boldsymbol{\alpha}_\tau} \underbrace{\{\mathrm{E}[u(c_\tau) + \delta V_{\tau+1}(W_{\tau+1}, s_{\tau+1}) \,|\, s_\tau]\}}_{L(c_\tau, \boldsymbol{\alpha}_\tau; W_\tau, s_\tau)} \tag{29}$$

subject to

$$(28)\, W_{\tau+1} = (W_\tau - c_\tau) \underbrace{(\boldsymbol{\alpha}'_\tau(\boldsymbol{R}_{\tau+1} - \mathbf{i}_N R^f_{\tau+1}) + R^f_{\tau+1})}_{R^W_{\tau+1}}, \quad c_\tau, \boldsymbol{\alpha}_\tau \in \mathcal{F}(s_\tau), \quad V_T(\cdot) = u(\cdot).$$

PROOF. Fix $\{c^{**}_\tau\}_{\tau=t}^{T-\nu-1}, \{\boldsymbol{\alpha}^{**}_\tau\}_{\tau=t}^{T-\nu-1}$ as any arbitrary choice of c_τ and $\boldsymbol{\alpha}_\tau$ for $\tau = t, \ldots, T-\nu-1$.

Let $\{c^*_\tau\}_{\tau=T-\nu}^{T-1}, \{\boldsymbol{\alpha}^*_\tau\}_{\tau=T-\nu}^{T-1}$ be the solution to

$$V_{T-\nu}(W_{T-\nu}, s_{T-\nu}) = \max_{\{c_\tau\}_{\tau=T-\nu}^{T-1}} \mathrm{E}\left[\sum_{\tau=T-\nu}^{T} \delta^{\tau-(T-\nu)} u(c_\tau) \,\middle|\, \begin{matrix} s_{T-\nu} \\ W_{T-\nu} \end{matrix} \right] \tag{30}$$

subject to (28) for $\tau = T-\nu,\dots,T-1$. Let $\{c_\tau^{**}\}_{\tau=T-\nu}^{T-1}, \{\boldsymbol{\alpha}_\tau^{**}\}_{\tau=T-\nu}^{T-1}$ be any other arbitrary choice. By definition

$$\mathrm{E}\left[\sum_{\tau=T-\nu}^{T}\delta^{\tau-(T-\nu)}u(c_\tau^*)\,\middle|\,\begin{matrix}s_{T-\nu}\\ W_{T-\nu}\end{matrix}\right] \geq \mathrm{E}\left[\sum_{\tau=T-\nu}^{T}\delta^{\tau-(T-\nu)}u(c_\tau^{**})\,\middle|\,\begin{matrix}s_{T-\nu}\\ W_{T-\nu}\end{matrix}\right]$$

which means using the law of iterated expectations,

$$\begin{aligned}\mathrm{E}&\left[\sum_{\tau=t}^{T-\nu-1}\delta^{\tau-t}u(c_\tau^{**})+\delta^{T-\nu-t}\left(\sum_{\tau=T-\nu}^{T}\delta^{\tau-(T-\nu)}u(c_\tau^*)\right)\,\middle|\,\begin{matrix}s_t\\ W_t\end{matrix}\right]\\ &\geq \mathrm{E}\left[\sum_{\tau=t}^{T-\nu-1}\delta^{\tau-t}u(c_\tau^{**})+\delta^{T-\nu-t}\left(\sum_{\tau=T-\nu}^{T}\delta^{\tau-(T-\nu)}u(c_\tau^{**})\right)\,\middle|\,\begin{matrix}s_t\\ W_t\end{matrix}\right].\end{aligned}$$

Since this holds for any $\{c_\tau^{**}\}_{\tau=t}^{T-\nu-1}$, it holds for $\{c_\tau^{*}\}_{\tau=t}^{T-\nu-1}$, the optimal solution to (27). Hence the solution to (30) is also the solution to (27) for $\tau = T-\nu,\dots,T-1$, so dynamic programming works for

$$u_t = u(c_t, c_{t-1},\dots,c_{t-s}).$$

which is habit persistence. But does it work for $u_t = u(c_t, c_{t+1},\dots,c_{t+s})$? □

It can be shown that if $u(\cdot)$ is increasing and concave, then $V_\tau(W_\tau, s_\tau)$ is increasing and concave in W_τ.

Envelope condition. We can show that

$$u'(c_\tau^*(W_\tau,s_\tau)) = V_{\tau,W}(W_\tau,s_\tau),$$

where $c_\tau^*(\cdot,\cdot)$ and $\boldsymbol{\alpha}_\tau^*(\cdot,\cdot)$ solve (29).

PROOF. The first-order condition for (29) wrt to c_τ is given by:

$$\frac{\partial L(c_\tau^*(W_\tau,s_\tau),\boldsymbol{\alpha}_\tau^*(W_\tau,s_\tau);W_\tau,s_\tau)}{\partial c_\tau}=0$$

$$u'(c_\tau^*(W_\tau,s_\tau)) - \delta\,\mathrm{E}[V_{\tau+1,W}(W_{\tau+1},s_{\tau+1}R_{\tau+1}^W\,|\,s_\tau] = 0. \tag{31}$$

The first-order condition for (29) wrt to $\boldsymbol{\alpha}_\tau$ is given by:

$$\frac{\partial L(c_\tau^*(W_\tau,s_\tau),\boldsymbol{\alpha}_\tau^*(W_\tau,s_\tau);W_\tau,s_\tau)}{\partial \boldsymbol{\alpha}_\tau}=\mathbf{0}$$

$$\mathrm{E}[V_{\tau+1,W}(W_{\tau+1},s_{\tau+1})(\boldsymbol{R}_{\tau+1}-R_{\tau+1}^f\mathbf{i}_N)\,|\,s_\tau]=\mathbf{0}.$$

Now

$$\begin{aligned}V_{\tau,W}(W_\tau,s_\tau) &= \frac{dL(c_\tau^*(W_\tau,s_\tau),\boldsymbol{\alpha}_\tau^*(W_\tau,s_\tau);W_\tau,s_\tau)}{dW_\tau}\\ &= \frac{\partial L(\cdot)}{\partial W_\tau} + \underbrace{\frac{\partial L(\cdot)}{\partial c_\tau)}}_{=0}\frac{\partial c^*}{\partial W_\tau} + \underbrace{\frac{\partial L(\cdot)}{\partial \boldsymbol{\alpha}_\tau}}_{=0}\frac{\partial \boldsymbol{\alpha}^*}{\partial W_\tau}\\ &= \delta\,\mathrm{E}[V_{\tau+1,W}(W_{\tau+1},s_{\tau+1})R_{\tau+1}^W\,|\,s_\tau].\end{aligned}$$

So using (31)

$$u'(c^*_\tau(W_\tau, s_\tau)) = V_{\tau,W}(W_\tau, s_\tau).$$

□

Infinite-lived investor. The problem at time t is

$$V_t(W_t, s_t) = \max \mathrm{E}\left[\sum_{\tau=0}^{\infty} \delta^\tau u(c_{t+\tau}) \mid s_t\right]$$

subject to (28) for $\tau = t, t+1, \ldots$. The problem at time $t+1$ is

$$V_{t+1}(W_{t+1}, s_{t+1}) = \max \mathrm{E}\left[\sum_{\tau=0}^{\infty} \delta^\tau u(c_{t+1+\tau}) \mid s_{t+1}\right]$$

subject to (28) for $\tau = t+1, t+2, \ldots$.

The problem at t looks the same as the problem at $t+1$, so if the solution to the right-hand side is finite, the Bellman equation becomes

$$V(W_\tau, s_\tau) = \max_{c_\tau, \boldsymbol{\alpha}_\tau} \{\mathrm{E}[u(c_\tau) + \delta V(W_{\tau+1}, s_{\tau+1}) \mid s_\tau]\}$$

subject to (28) for $\tau = t, t+1, \ldots$.

7.6. Campbell–Viceira approximate solution for the multi-period agent's problem

Consider a multi-period investor who solves the following problem:

$$V_t(W_t, x_t) = \max_{\{C_\tau\}_{\tau=t}^T \{\alpha_\tau\}_{\tau=t}^T}, \left\{\mathrm{E}_t\left[\sum_{\tau=t}^{T} \delta^{\tau-t} \frac{C_\tau^{1-\gamma}}{1-\gamma}\right]\right\}, \quad C_\tau, \alpha_\tau \in \mathcal{F}_\tau \forall \tau.$$

s.t.

$$W_{\tau+1} = (W_\tau - C_\tau) R_{W,\tau+1} \tag{32}$$

and

$$R_{W,\tau+1} = \alpha_t (R_{1,\tau+1} - R_f) + R_f. \tag{33}$$

Let $r_{1,\tau+1} = \ln(R_{1,\tau+1})$, $r_{W,\tau+1} = \ln(R_{W,\tau+1})$, $r_f = \ln(R_f)$, $c_{\tau+1} = \ln(C_{\tau+1})$ and $w_{\tau+1} = \ln(W_{\tau+1})$. The return generating process satisfies:

$$\mathrm{E}_\tau[r_{1,\tau+1}] - r_f = x_\tau,$$
$$x_{\tau+1} = \mu + \varphi(x_\tau - \mu) + \eta_{\tau+1},$$

where $0 < \varphi < 1$ and μ are constants and $\eta_{\tau+1}$ is normal i.i.d. white noise with zero mean and standard deviation of σ_η. The conditional mean of $(\mathrm{E}_\tau[r_{1,\tau+1}] - r_f)$ is given by the state variable x_τ which evolves through time as an AR(1) process. Moreover, $r_{1,\tau+1}$ is conditionally homoskedastic. Letting $u_{\tau+1} = r_{1,\tau+1} - \mathrm{E}_\tau[r_{1,\tau+1}]$, assume that

$$\mathrm{var}_\tau[u_{\tau+1}] = \sigma_u^2,$$
$$\mathrm{cov}_\tau[u_{\tau+1}, \eta_{\tau+1}] = \sigma_{u\eta}.$$

The Euler equation for $R_{i,t+1}$ that comes out of the first order condition for this investor's problem for $i = 1, W$ and f says

$$1 = \mathrm{E}_t \left[\delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{i,t+1} \right]. \tag{34}$$

Let $V_{x,y,t} \equiv \mathrm{cov}_t[x_{t+1}, y_{t+1}]$. Assume $r_{1,\tau+1}$ and $c_{\tau+1} \equiv \ln C_{\tau+1}$ are multivariate normal conditional on time-τ information. Then (34) implies that for $i = W$,

$$\begin{aligned} 0 &= \ln \delta - \gamma \, \mathrm{E}_t[\Delta c_{t+1}] + \mathrm{E}_t[r_{1,t+1}] \\ &\qquad + \tfrac{1}{2}(\gamma^2 \sigma_t^2[\Delta c_{t+1}] + \sigma_t^2[r_{1,t+1}] - 2\gamma \sigma_t[\Delta c_{t+1}, r_{1,t+1}]), \\ \mathrm{E}_t[r_{1,t+1}] &= -\ln \delta + \gamma \, \mathrm{E}_t[\Delta c_{t+1}] \\ &\qquad - \tfrac{1}{2}\gamma^2 \sigma_t^2[\Delta c_{t+1}] - \tfrac{1}{2}\sigma_t^2[r_{1,t+1}] + \gamma \sigma_t[\Delta c_{t+1}, r_{1,t+1}], \\ r_f &= -\ln \delta + \delta \, \mathrm{E}_t \, \Sigma c_{t+1} - \tfrac{1}{2}\sigma^2 \sigma_t^2[\Delta c_{t+1}], \end{aligned}$$

and so

$$\mathrm{E}_t[r_{1,t+1}] - r_f + \tfrac{1}{2} V_{1,1,t}^2 = \gamma V_{1,c,t}. \tag{35}$$

But $c_{\tau+1}$ and $r_{1,\tau+1}$ are not multivariate normal conditional on time-τ information even if $r_{1,\tau+1}$ is conditionally normal. Even if $r_{1,t+1}$ and c_{t+1} are not multivariate normal conditional on the time-t information, it is possible to obtain (35) using log-linearizations.

Let $\Delta w_{\tau+1} = w_{\tau+1} - w_\tau$. We can use a log-linearization of the budget constraint (32) to obtain an approximate expression for $\Delta w_{\tau+1}$ that is linear in $r_{W,\tau+1}$ and $(c_\tau - w_\tau)$. We log-linearize the budget constraint by a first order Taylor expansion of $c_t - w_t$ around $\bar{c} - \bar{w}$. Letting $\rho = 1 - \exp(\bar{c} - \bar{w})$,

$$\begin{aligned} W_{t+1} &= R_{W,t+1}(W_t - C_t), \\ W_{t+1}/W_t &= R_{W,t+1}\left(1 - \frac{C_t}{W_t}\right) \\ \Delta w_{t+1} &= r_{W,t+1} + \ln(1 - \exp(c_t - w_t)) \\ &= r_{W,t+1} + \ln(1 - \exp(\bar{c} - \bar{w})) + \frac{-\exp(\bar{c} - \bar{w})}{1 - \exp(\bar{c} - \bar{w})}(c_t - w_t - (\bar{c} - \bar{w})) \\ &= r_{W,t+1} + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) + \left(\ln(\rho) - \left(1 - \frac{1}{\rho}\right)\ln(1 - \rho)\right). \end{aligned} \tag{36}$$

We can log-linearize the portfolio return equation (33) by a first order Taylor expansion of $r_{W,t+1}$ with respect to $r_{1,t+1}$ around zero and r_f around zero:

$$\begin{aligned} R_{W,t+1} &= \alpha_t(R_{1,t+1} - R_f) + R_f, \\ r_{W,t+1} &= \ln(\alpha_t \exp(r_{1,t+1}) - \alpha_t \exp(r_f) + \exp(r_f)) \\ &= \ln(\alpha_t \exp(0) - \alpha_t \exp(0) + \exp(0)) \qquad (37) \\ &\quad + \frac{\alpha_t \exp(0)}{\alpha_t \exp(0) - \alpha_t \exp(0) + \exp(0)}(r_{1,t+1} - 0) \\ &\quad + \frac{(1-\alpha_t) \exp(0)}{\alpha_t \exp(0) - \alpha_t \exp(0) + \exp(0)}(r_f - 0) \\ &= \alpha_t(r_{1,t+1} - r_f) + r_f. \qquad (38) \end{aligned}$$

We can use the approximate expressions derived in (36), (38) and (35) to show that the following approximate relation holds:

$$\alpha_t = \frac{\mathrm{E}_t[r_{1,t+1}] - r_f + \frac{1}{2}\sigma_u^2}{\gamma\sigma_u^2} - \frac{\mathrm{cov}_t[r_{1,t+1}, c_{t+1} - w_{t+1}]}{\sigma_u^2}. \qquad (39)$$

Combine (36) and (38) to get

$$\Delta w_{t+1} \approx \alpha_t(r_{1,t+1} - r_f) + r_f + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) + k. \qquad (40)$$

Using (40) and

$$\Delta c_{t+1} = (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}, \qquad (41)$$

we can show that $V_{1c,t}$ can be written as a function of $V_{1,c-w,t}$, α_t and $V_{1,1,t}$:

$$\begin{aligned} V_{1,c,t} &= \mathrm{cov}_t[r_{1,t+1}, \Delta c_{t+1}] \\ &= \mathrm{cov}_t[r_{1,t+1}, c_{t+1} - w_{t+1}] - \mathrm{cov}_t[r_{1,t+1}, c_t - w_t] \\ &\quad + \mathrm{cov}_t[r_{1,t+1}, \alpha_t r_{1,t+1} + (1-\alpha_t)r_f + (1 - \tfrac{1}{\rho})(c_t - w_t) + k] \\ &= V_{1,c-w,t} + \alpha_t V_{1,1,t}. \end{aligned}$$

Using (35),

$$\mathrm{E}_t[r_{1,t+1}] - r_f + \tfrac{1}{2}V_{1,1,t} = \gamma(V_{1,c-w,t} + \alpha_t V_{1,1,t}),$$

$$\alpha_t = \frac{\mathrm{E}_t[r_{1,t+1}] - r_f + \frac{1}{2}V_{1,1,t}}{V_{1,1,t}}\frac{1}{\gamma} - \frac{V_{1,c-w,t}}{V_{1,1,t}}$$

as required.

Suppose that the investor's life ended at time-$(t+1)$: i.e., $t = T-1$. How would the formula for α_t in (39) change? If the investor's life ended at time-$(t+1)$, then $C_{t+1} = W_{t+1}$ and so $c_{t+1} - w_{t+1} = 0$ and $V_{1,c-w,t} = 0$. Consequently, the the single-period agent's risky-asset allocation, which is known as the myopic demand, is given by:

$$\alpha_t = \frac{\mathrm{E}_t[r_{1,t+1}] - r_f + \frac{1}{2}V_{1,1,t}}{V_{1,1,t}}\frac{1}{\gamma}$$

The difference between the the risky-asset allocations by a multi-period agent and a single-period agent, which is given by $-\frac{V_{1,c-w,t}}{V_{1,1,t}}$, is known as the hedging demand and converges to 0 as t converges to $T-1$. Suppose $\sigma_{u\eta}=0$. Since x_t is the only state variable for the agent's problem and recalling that the power utility problem is homogeneous in wealth, we know that $c_{t+1}-w_{t+1}$ only depends on x_{t+1}. Thus, if $\sigma_{u\eta}=0$, then $V_{1,c-w,t}=0$ and the hedging demand is zero. So the risky-asset return from t to $t+1$ needs to be $\mathcal{F}_t$-conditionally correlated with the state variable at $t+1$, x_{t+1}, for the hedging demand to be non-zero.

It is possible to obtain an explicit expression for α_t as a function of x_t. For simplicity, suppose that the agent is infinitely lived: i.e., $T=\infty$. It is possible to show that the solution for c_t satisfies

$$c_t - w_t = b_0 + b_1 x_t + b_2 x_t^2, \tag{42}$$

where b_0, b_1 and b_2 are constants. We can use (39) and the result

$$x_{\tau+1}^2 - \mathrm{E}_\tau[x_{\tau+1}^2] = (\eta_{\tau+1}^2 - \sigma_\eta^2) + (2\mu(1-\varphi) + 2\varphi x_\tau)\eta_{\tau+1}$$

to show that the solution for α_t is linear in x_t:

$$\alpha_t = a_0 + a_1 x_t.$$

We can obtain expressions for a_0 and a_1 in terms of γ, σ_u^2, $\sigma_{u\eta}$, μ, φ, b_1 and b_2. Taking (42) as given and noting that

$$x_{t+1}^2 - \mathrm{E}_t[x_{t+1}^2] = (\eta_{t+1}^2 - \sigma_\eta^2) + (2\mu(1-\varphi) + 2\varphi x_t)\eta_{t+1},$$

it follows that

$$\begin{aligned} V_{1,c-w,t} &= \mathrm{cov}_t[r_{1,t+1}, c_{t+1} - w_{t+1}] \\ &= \mathrm{cov}[u_{t+1}, b_1(x_{t+1} - \mathrm{E}_t\, x_{t+1}) \\ &\qquad + b_2(\eta_{t+1}^2 - \sigma_\eta^2 + (2\mu(1-\varphi) + 2\varphi x_t)\eta_{t+1}] \\ &= \mathrm{cov}[u_{t+1}, b_1\eta_{t+1} + b_2(2\mu(1-\varphi)) + 2\varphi x_t)\eta_{t+1}] \\ &= (b_1 + b_2(2\mu(1-\varphi) + 2\varphi x_t))\sigma_{u\eta}, \end{aligned}$$

and so

$$\begin{aligned} \alpha_t &= \frac{x_\tau + \frac{1}{2}\sigma_u^2}{\sigma_u^2}\frac{1}{\gamma} - \frac{1}{\sigma_u^2}(b_1 + b_2(2\mu(1-\varphi) + 2\varphi x_t)\sigma_{u\eta} \\ &= \underbrace{\frac{1}{2\gamma} - \frac{1}{\sigma_u^2} - \frac{1}{\sigma_u^2}b_2 2\mu(1-\varphi)\sigma_{u\eta}}_{a_0} \\ &\qquad + \underbrace{\left(\frac{1}{\gamma\sigma_u^2} - \frac{1}{\sigma_u^2}b_2 2\varphi\sigma_{u\eta}\right)}_{a_1} x_t \end{aligned}$$

as required.

It is worth noting that the analysis can be extended to:

- multiple risky assets.