it follows that idiosyncratic shocks with the same distribution across aggregate states do not affect the equity premium and cannot help explain the equity premium puzzle.

Importance of a continuum of agents. How important is the assumption of a continuum of agents for the results obtained above? The answer is not very important since it is possible to describe an economy with a finite number of agents and the same parameters, μ_G, μ_B, φ_G, φ_B and γ, which has identical prices for equity and the riskless asset (as in the economy above) whenever the parameter values for μ_G, μ_B, φ_G, φ_B and γ are the same (as in the economy above).

Consider an economy with two agents each with the same utility function as in a continuous economy. Each is endowed with consumption at time 0 normalized to 1. There are two possible states at time 1, G and B, that are equally likely. In state G, one agent receives $\mu_G(1+\varphi_G)$ and the other receives $\mu_G(1-\varphi_G)$. Each agent has a 50% chance of receiving $\mu_G(1+\varphi_G)$. In state B, one agent receives $\mu_B(1+\varphi_B)$ and the other receives $\mu_B(1-\varphi_B)$. Again, each agent has a 50% chance of receiving $\mu_B(1+\varphi_B)$. There are the same two assets as in the economy with the continuum of agents. Since the first order conditions of the agents and the aggregate quantities are the same as those in the economy with a continuum of agents, it follows that the prices in this economy are the same as those in that economy.

Implications for the equity premium puzzle. To summarize, the model developed in this subsection shows the following:

- idiosyncratic uninsurable labor income risk can help the equity premium puzzle if it's concentrated in bad states.
- idiosyncratic uninsurable labor income risk does not effect pricing if it's independent of the state of the economy.
- the model's assumption of a continuum of agents is likely not critical for the first two results.

13.3. Epstein–Zin preferences

CRRA utility. Consider the problem faced an infinitely-lived agent with CRRA preferences:

$$\max_{\{c_\tau,\boldsymbol{\alpha}_\tau\}_{\tau=t}^{\infty}} \mathrm{E}\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}(1-\delta)\frac{C_\tau^{1-\gamma}}{1-\gamma}\,\middle|\,\mathcal{F}_t\right],$$

subject to

$$C_\tau,\boldsymbol{\alpha}_\tau\in\mathcal{F}_\tau,\ W_{\tau+1}=(W_\tau-C_\tau)(\boldsymbol{\alpha}_\tau(\mathbf{R}_{\tau+1}-R_{f,\tau+1}\mathbf{i}_N)+R_{f,\tau+1}). \tag{78}$$

where $R_{f,\tau+1}$ is the riskfree rate from τ to $\tau+1$ and $\mathbf{R}_{\tau+1}$ is a $N\times 1$ vector of risky asset returns from τ to $\tau+1$.

The Bellman equation has the form

$$V(W_\tau,\mathcal{F}_\tau)=\max_{c_\tau,\boldsymbol{\alpha}_\tau}\left\{(1-\delta)\frac{C_\tau^{1-\gamma}}{1-\gamma}+\delta\,\mathrm{E}[V(W_{\tau+1},\mathcal{F}_{\tau+1})\,|\,\mathcal{F}_\tau]\right\}\quad\text{s.t. (78).}$$

We know that the solution satisfies

$$C_\tau=\varphi(\mathcal{F}_\tau)W_\tau\quad\text{and}\quad\boldsymbol{\alpha}_\tau=\boldsymbol{\alpha}(\mathcal{F}_\tau), \tag{79}$$

and we know the form of the value function is

$$a(\mathcal{F}_\tau)W_\tau^{1-\gamma} = V(W_\tau, \mathcal{F}_\tau).$$

Define

$$J(W_\tau, \mathcal{F}_\tau) = [(1-\gamma)V(W_\tau, \mathcal{F}_\tau)]^{\frac{1}{1-\gamma}}.$$

Then it follows that

$$J(W_\tau, \mathcal{F}_\tau) = \phi(\mathcal{F}_\tau)W_\tau$$

and the Bellman equation can be rewritten:

$$J(W_\tau, \mathcal{F}_\tau) = \max_{c_\tau, \boldsymbol{\alpha}_\tau} \left[(1-\delta)C_\tau^{1-\gamma} + \delta\, \mathrm{E}[J(W_{t+1}, \mathcal{F}_{\tau+1})^{1-\gamma} \mid \mathcal{F}_\tau]\right]^{\frac{1}{1-\gamma}} \quad \text{s.t. (78).}$$

The first-order condition is

$$\forall i = f, 1, \ldots, N : 1 = \mathrm{E}_t \left[\delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{i,t+1} \,\middle|\, \mathcal{F}_\tau \right].$$

A more general specification. Consider the following more general specification for the Bellman equation:

$$J(W_\tau, \mathcal{F}_\tau) = \max_{c_\tau, \boldsymbol{\alpha}_\tau} \left[(1-\delta)C_\tau^{\rho} + \delta(\mathrm{E}[J(W_{\tau+1}, \mathcal{F}_{\tau+1})^{\alpha} \mid \mathcal{F}_\tau])^{\rho/\alpha}\right]^{1/\rho} \quad \text{s.t. (78).} \tag{80}$$

This is the Bellman equation for an infinitely-lived investor with Epstein-Zin preferences. Notice that when $\rho = \alpha = 1-\gamma$, this specification collapses down to the rewritten Bellman equation for the CRRA case. So power preferences can be regarded as a special case of Epstein-Zin preferences. It can be shown that for Epstein-Zin prefernces, $\alpha = 1-\gamma$, where γ is relative risk aversion, and $\rho = 1 - 1/\psi$, where ψ is the elasticity of intertemporal substitution. So $\alpha = \rho \Leftrightarrow \gamma = 1/\psi$ as expected.

Preferences for the timing of the resolution of uncertainty. An investor with Epstein-Zin preferences cares about the timing of the resolution of uncertainty:

- $\alpha < \rho$ or $\gamma > 1/\psi$ means early resolution of uncertainty is preferred
- $\alpha > \rho$ or $\gamma < 1/\psi$ means late resolution of uncertainty is preferred.

Here is an example to illustrate this. Consider an investor with Epstein-Zin preferences who consumes at three dates: 0, 1, and 2. Hold C_0 and C_1 constant and let

$$C_2 = \begin{cases} C^u & \text{with probability } \frac{1}{2}, \\ C^d & \text{with probability } \frac{1}{2}. \end{cases}$$

Suppose C_2 is not known until time 2. Then the agent's value function at times 1 and 2 can be written as

$$J_1 = \left[(1-\delta)C_1^\rho + \delta\left[\tfrac{1}{2}((C^u)^\alpha + (C^d)^\alpha)\right]^{\rho/\alpha}\right]^{1/\rho} \qquad \text{and}$$

$$J_0 = \left[(1-\delta)C_0^\rho + \delta(J_1^\alpha)^{\rho/\alpha}\right]^{1/\rho}$$

$$= \left[(1-\delta)C_0^\rho + \delta\left[(1-\delta)C_1^\rho + \delta\big(\tfrac{1}{2}((C^u)^\alpha + (C^d)^\alpha)\big)^{\rho/\alpha}\right]\right]^{1/\rho},$$

respectively.

Suppose instead that C_2 is known at time 1. Then the agent's value function at time 1 depends on consumption at time 2:

$$\hat{J}_1(C^u) = \left[(1-\delta)C_1^\rho + \delta[(C^u)^\alpha]^{\rho/\alpha}\right]^{1/\rho}$$

$$= \left[(1-\delta)C_1^\rho + \delta(C^u)^\rho\right]^{1/\rho} \qquad \text{and}$$

$$\hat{J}_1(C^d) = \left[(1-\delta)C_1^\rho + \delta(C^d)^\rho\right]^{1/\rho},$$

and so the agent's value function at time 0 is given by

$$\hat{J}_0 = \left[(1-\delta)C_0^\rho + \delta\big(\tfrac{1}{2}(\hat{J}_1(C^u)^\alpha + \hat{J}_1(C^d)^\alpha\big)^{\rho/\alpha}\right]^{1/\rho}$$

$$= \left[(1-\delta)C_0^\rho + \delta\big(\tfrac{1}{2}\big([(1-\delta)C_1^\rho + \delta(C^u)^\rho]^{\alpha/\rho} + [(1-\delta)C_1^\rho + \delta(C^d)^\rho]^{\alpha/\rho}\big)\big)^{\rho/\alpha}\right]^{1/\rho}.$$

Now $\hat{J}_0 > J_0$ iff

$$\frac{1}{\delta}(\hat{J}_0^\rho - (1-\delta)C_0^\rho)\operatorname{sign}(\rho) > \frac{1}{\delta}(J_0^\rho - (1-\delta)C_0^\rho)\operatorname{sign}(\rho)$$

$\Leftrightarrow$

$$\left(\frac{1}{2}^{\rho/\alpha}([(1-\delta)C_1^\rho + \delta(C^u)^\rho]^{\alpha/\rho} + [(1-\delta)C_1^\rho + \delta(C^d)^\rho]^{\alpha/\rho})^{\rho/\alpha}\right)\operatorname{sign}(\rho)$$
$$> \left((1-\delta)C_1^\rho + \delta\frac{1}{2}^{\rho/\alpha}((C^u)^\alpha + (C^d)^\alpha)^{\rho/\alpha}\right)\operatorname{sign}(\rho)$$

$\Leftrightarrow$

$$\operatorname{sign}(\rho)\Big(\mathrm{E}\left[(k_1 + \delta C_2^\rho)^{\alpha/\rho}\right]\Big)^{\rho/\alpha} > \big(k_1 + \delta(\mathrm{E}[C_2^\alpha])^{\rho/\alpha}\big)\operatorname{sign}(\rho), \quad k_1 > 0.$$

$\Leftrightarrow$

$$\operatorname{sign}(\rho)\operatorname{sign}\left(\frac{\rho}{\alpha}\right)\mathrm{E}[(k_1 + \delta(C_2^\alpha)^{\rho/\alpha})^{\alpha/\rho}] > (k_1 + \delta\,\mathrm{E}[C_2^\alpha]^{\rho/\alpha})^{\alpha/\rho}\operatorname{sign}(\rho)\operatorname{sign}\left(\frac{\rho}{\alpha}\right).$$

Define

$$f(C_2^\alpha) = (k_1 + \delta(C_2^\alpha)^{\rho/\alpha})^{\alpha/\rho}.$$

Now

$$\begin{aligned} f'(C_2^\alpha) &= (k_1 + \delta(C_2^\alpha)^{\rho/\alpha})^{\alpha/\rho-1}\frac{\alpha}{\rho}\delta(C_2^\alpha)^{\rho/\alpha-1}\frac{\rho}{\alpha} \\ &= (k_1(C_2^\alpha)^{-\rho/\alpha} + \delta)^{\alpha/\rho-1}, \qquad \text{and} \\ f''(C_2^\alpha) &= \left(\frac{\alpha}{\rho}-1\right)(k_1(C_2^\alpha)^{-\rho/\alpha}+\delta)^{\alpha/\rho-2}k_1\left(-\frac{\rho}{\alpha}\right)(C_2^\alpha)^{-\rho/\alpha-1}, \end{aligned}$$

since

$$\frac{\frac{\rho}{\alpha}-1}{\frac{\alpha}{\rho}-1} = \frac{\frac{\rho-\alpha}{\alpha}}{\frac{\alpha-\rho}{\alpha}} = -\frac{\rho}{\alpha}.$$

Jensen's inequality says that for an arbitrary twice-continuously differentiable function $\hat{f}(.)$,

$$\mathrm{E}[\hat{f}(q)] \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \hat{f}(\mathrm{E}[q]) \quad \Leftrightarrow \quad \hat{f}''(\cdot) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0.$$

Now define

$$\hat{f}(C_2^\alpha) = \operatorname{sign}(\rho)\operatorname{sign}\left(\frac{\rho}{\alpha}\right) f(C_2^\alpha).$$

Then

$$\hat{f}''(C_2^\alpha) = \underbrace{\operatorname{sign}(\rho)\frac{1}{\rho}}_{+}(\alpha-\rho)\underbrace{\operatorname{sign}\left(\frac{\rho}{\alpha}\right)\left(-\frac{\rho}{\alpha}\right)}_{-}\underbrace{(k_1(C_2^\alpha)^{-\rho/\alpha}+\delta)^{\alpha/\rho-2}}_{+}\underbrace{k_1(C_2^\alpha)^{-\rho/\alpha-1}}_{+}.$$

So it follows that

$$\hat{f}''(\cdot)\begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \Leftrightarrow \alpha \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \rho \Leftrightarrow \mathrm{E}[\hat{f}(C_2^\alpha)] \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \hat{f}(\mathrm{E}[C_2^\alpha]) \Leftrightarrow \hat{J}_0 \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} J_0$$

$\Leftrightarrow$

$$\text{agent} \begin{cases} \text{prefers early resolution of uncertainty.} \\ \text{does not care about the timing of uncertainty resolution.} \\ \text{prefers late resolution of uncertainty.} \end{cases}$$

This example illustrates how the agent's preference for the timing of the resolution of uncertainty depends on the relative magnitude of α and ρ.

First-order conditions and solution. Suppose the value function can be written

$$J(W_{t+1}, \mathcal{F}_{t+1}) = \phi(\mathcal{F}_{t+1}) W_{t+1}$$

and the solution satisfies (79), then the Bellman equation for Epstein-Zin preferences (80) can be written

$$\phi(\mathcal{F}_t) W_t = \max_{\varphi(\mathcal{F}_t), \boldsymbol{\alpha}(\mathcal{F}_t)} \Big[(1-\delta)\varphi(\mathcal{F}_t)^\rho + \delta(1-\varphi(\mathcal{F}_t))^\rho \underbrace{(E[(\phi(\mathcal{F}_{t+1}) R_{W,t+1})^\alpha \mid \mathcal{F}_t])^{\rho/\alpha}}_{\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho} \Big]^{1/\rho} W_t, \tag{81}$$

since

$$\text{s.t.} \quad W_{t+1} = W_t(1-\varphi(\mathcal{F}_t)) \underbrace{(\boldsymbol{\alpha}(\mathcal{F}_t)'(\mathbf{R}_{t+1} - R^f_{t+1}\mathbf{i}_N) + R^f_{t+1})}_{R_{W,t+1}} .$$

The first-order condition for this problem with respect to $\boldsymbol{\alpha}_t$ is given by

$$\mathrm{E}[\varphi(\mathcal{F}_{t+1})^\alpha (R_{W,t+1})^{\alpha-1}(\mathbf{R}_{t+1} - R^f_{t+1}\mathbf{i}_N) \mid \mathcal{F}_t] = \mathbf{0}, \tag{82}$$

which shows that $\boldsymbol{\alpha}_\tau$ does not depend on W_τ as conjectured. The first-order condition with respect to φ is

$$(1-\delta)\rho\varphi(\mathcal{F}_t)^{\rho-1} - \delta\rho(1-\varphi(\mathcal{F}_t))^{\rho-1}\Omega(\mathcal{F}_t;\boldsymbol{\alpha}_t)^\rho = 0, \tag{83}$$

which can be rewritten

$$\begin{aligned}
&(1-\delta)\varphi(\mathcal{F}_t)^{\rho-1} = \delta(1-\varphi(\mathcal{F}_t))^{\rho-1}\Omega(\mathcal{F}_t;\boldsymbol{\alpha}_t)^\rho \\
&\Leftrightarrow \left(\frac{1-\delta}{\delta}\right)^{\frac{1}{\rho-1}} \varphi(\mathcal{F}_t) = (1-\varphi(\mathcal{F}_t))\Omega(\mathcal{F}_t;\boldsymbol{\alpha}_t)^{\frac{\rho}{\rho-1}} \\
&\Leftrightarrow \left[\left(\frac{\delta}{1-\delta}\right)^{\frac{1}{1-\rho}} + \Omega(\mathcal{F}_t;\boldsymbol{\alpha})^{\frac{\rho}{\rho-1}}\right]\varphi(\mathcal{F}_t) = \Omega(\mathcal{F}_t;\boldsymbol{\alpha}_t)^{\frac{\rho}{\rho-1}}
\end{aligned}$$

$$\Leftrightarrow$$

$$\varphi(\mathcal{F}_t) = \frac{1}{\left[\frac{\delta}{1-\delta}\Omega(\mathcal{F}_t;\boldsymbol{\alpha}_t)^\rho\right]^{\frac{1}{1-\rho}} + 1}. \tag{84}$$

This expression shows that $\varphi(\mathcal{F}_t)$ does not depend on wealth W_t as conjectured.

Now to verify the conjectured form of the value function we first substitute (84) into the Bellman equation (81) to obtain

$$\begin{aligned}
J(W_t, \mathcal{F}_t) &= \left[(1-\delta)\varphi(\mathcal{F}_t)^\rho W_t^\rho + \delta(1-\varphi(\mathcal{F}_t))^\rho W_t^\rho \Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho\right]^{\frac{1}{\rho}} \\
&= \left[(1-\delta)\varphi(\mathcal{F}_t)^\rho + \delta(1-\varphi(\mathcal{F}_t))^\rho \Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho\right]^{1/\rho} W_t \\
&= \left(\frac{(1-\delta) + \delta\frac{(1-\delta)}{\delta}\left[\frac{\delta}{1-\delta}\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho\right]^{\frac{\rho}{1-\rho}+1}}{\left(\left[\dfrac{\delta}{1-\delta}\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho\right]^{\frac{1}{1-\rho}}+1\right)^\rho}\right)^{1/\rho} W_t \\
&= (1-\delta)^{1/\rho}\left(\frac{1+\left(\dfrac{\delta}{1-\delta}\right)^{1+\frac{\rho}{1-\rho}}(\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho)^{\frac{\rho}{1-\rho}+1}}{\left(\left[\dfrac{\delta}{1-\delta}\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho\right]^{\frac{1}{1-\rho}}+1\right)^\rho}\right)^{1/\rho} W_t \\
&= (1-\delta)^{1/\rho}\varphi(\mathcal{F}_t)^{\frac{\rho-1}{\rho}} W_t,
\end{aligned}$$

as conjectured, with

$$\phi(\mathcal{F}_t) = (1-\delta)^{1/\rho}\varphi(\mathcal{F}_t)^{\frac{\rho-1}{\rho}}. \tag{85}$$

We have just shown that the form of the value function and the solution are in fact as conjectured.

Next we show how to obtain a representation of the first order conditions for $\boldsymbol{\alpha}$ and c that is often used in the literature and that is used below to obtain expressions for conditional expected log asset returns. First, equation (83), which is the first order condition with respect to φ, can be written as

$$(1-\delta)\varphi(\mathcal{F}_t)^{\rho-1} = \delta(1-\varphi(\mathcal{F}_t))^{\rho-1}\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho,$$

which can be rewritten as

$$\varphi(\mathcal{F}_t)^{(\rho-1)\frac{\alpha}{\rho}} = \left(\frac{\delta}{1-\delta}\right)^{\alpha/\rho}(1-\varphi(\mathcal{F}_t))^{(\rho-1)\frac{\alpha}{\rho}}\,\mathrm{E}[\phi(\mathcal{F}_{t+1}^\alpha)(R_{W,t+1})^\alpha \mid \mathcal{F}_t]. \tag{86}$$

Then take equation (82), which is the first order condition for $\boldsymbol{\alpha}$, and premultiply by $\boldsymbol{\alpha}(\mathcal{F}_t)$ to obtain

$$\mathrm{E}[\phi(\mathcal{F}_{t+1})^\alpha(R_{W,t+1})^{\alpha-1}(R_{W,t+1} - R_{t+1}^f) \mid \mathcal{F}_t] = 0. \tag{87}$$

Now (82), (87) and (86) imply

$$\begin{aligned}
&\varphi(\mathcal{F}_t)^{(\rho-1)\frac{\alpha}{\rho}} \\
&\quad = \left(\frac{\delta}{1-\delta}\right)^{\frac{\alpha}{\rho}}(1-\varphi(\mathcal{F}_t))^{(\rho-1)\frac{\alpha}{\rho}}\,\mathrm{E}[\phi(\mathcal{F}_{t+1})^\alpha(R_{W,t+1})^{\alpha-1}R_{i,t+1} \mid \mathcal{F}_t], \quad i = f, 1, \ldots, N.
\end{aligned} \tag{88}$$

Finally substitute (85) into (88) to obtain

$$\underbrace{\varphi(\mathcal{F}_t)^{(\rho-1)\frac{\alpha}{\rho}}}_{C_t/W_t} = \left(\frac{\delta}{1-\delta}\right)^{\frac{\alpha}{\rho}} \mathrm{E}\Big[\underbrace{(1-\varphi(\mathcal{F}_t))^{(\rho-1)\frac{\alpha}{\rho}}\varphi(\mathcal{F}_{t+1})^{\frac{\rho-1}{\rho}\alpha}(R_{W,t+1})^{\frac{\rho-1}{\rho}\alpha}}_{C_{t+1}/W_t}$$
$$(1-\delta)^{\frac{\alpha}{\rho}}(R_{W,t+1})^{\frac{1-\rho}{\rho}\alpha}(R_{W,t+1})^{\alpha-1}R_{i,t+1} \,\Big|\, \mathcal{F}_t\Big]$$

$\Rightarrow$

$$1 = \delta^{\frac{\alpha}{\rho}}\,\mathrm{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\rho-1}{\rho}\alpha}(R_{W,t+1})^{\frac{\alpha-\rho}{\rho}}R_{i,t+1}\,\middle|\,\mathcal{F}_t\right], \quad i = f, 1, \ldots, N. \tag{89}$$

Notice that this expression collapses to the CRRA first order condition when $\rho = \alpha$.

Log risk free rate and expected log risky asset return. Suppose

$$g_{t+1} \equiv \ln\left(\frac{C_{t+1}}{C_t}\right), \quad r_{W,t+1} \equiv \ln(R_{W,t+1}) \quad \text{and} \quad r_{i,t+1} \equiv \ln(R_{i,t+1})$$

are conditionally distributed

$$\text{Normal}\left(\begin{bmatrix}\mu_{g,t}\\ \mu_{W,t}\\ \mu_{i,t}\end{bmatrix}, \begin{bmatrix}\sigma^2_{g,t} & \sigma_{gW,t} & \sigma_{gi,t}\\ \sigma_{Wg,t} & \sigma^2_{W,t} & \sigma_{Wi,t}\\ \sigma_{ig,t} & \sigma_{iW,t} & \sigma^2_{i,t}\end{bmatrix}\right),$$

and let $r_{f,t+1} \equiv \ln(R_{f,t+1})$.

With these assumptions and definitions, it is possible to obtain closed-form expressions for $r_{f,t+1}$ and

$$\ln(\mathrm{E}_t[R_{i,t+1}]/R_{f,t+1}) = \mathrm{E}_t[r_{i,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma^2_{i,t}.$$

Define

$$m_{t+1} = \theta\ln\delta - \frac{\theta}{\psi}g_{t+1} + (\theta-1)r_{W,t+1},$$

where

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}.$$

Using the first order condition (89) derived above, we know that $\exp\{m_{t+1}\}$ is the agent's marginal rate of substitution which can be used as an sdf. So

$$\mathrm{E}_t[\exp\{m_{t+1} + r_{i,t+1}\}] = 1,$$

$\Leftrightarrow$

$$\exp\left\{\mathrm{E}_t[m_{t+1} + r_{i,t+1}] + \tfrac{1}{2}\sigma^2_t[m_{t+1} + r_{i,t+1}]\right\} = 1,$$

$\Leftrightarrow$

$$\mathrm{E}_t[m_{t+1} + r_{i,t+1}] + \tfrac{1}{2}\sigma^2_t[m_{t+1}] + \tfrac{1}{2}\sigma^2_t[r_{i,t+1}] + \sigma_t[m_{t+1}, r_{i,t+1}] = 0. \tag{90}$$

Since (90) implies that

$$-\mathrm{E}_t[m_{t+1}] - \tfrac{1}{2}\sigma_t^2[m_{t+1}] = r_{f,t+1},$$

it follows that

$$\mathrm{E}_t[r_{i,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma_t^2[r_{i,t+1}] = -\sigma_t[m_{t+1}, r_{i,t+1}].$$

And finally, since the definition of m_{t+1} implies that

$$\sigma_t[m_{t+1}, r_{i,t+1}] = -\frac{\theta}{\psi}\sigma_{gi,t} + (\theta - 1)\sigma_{Wi,t},$$

it follows that

$$\mathrm{E}_t[r_{i,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma_{i,t}^2 = \frac{\theta}{\psi}\sigma_{gi,t} + (1 - \theta)\sigma_{Wi,t}. \tag{91}$$

Setting $i = W$ in equation (91) gives us the following expression for the expected log return on the agent's wealth portfolio:

$$\mathrm{E}_t[r_{W,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma_{W,t}^2 = \frac{\theta}{\psi}\sigma_{gW,t} + (1 - \theta)\sigma_{W,t}^2. \tag{92}$$

Using the definition of θ, it follows that

$$1 - \theta = \frac{\gamma - \frac{1}{\psi}}{1 - \frac{1}{\psi}}.$$

So if $\psi > 1$, then a preference for early resolution of uncertainty (i.e., $\gamma > \frac{1}{\psi}$) implies that the agent dislikes the volatility associated with the log excess return on her wealth portfolio and so requires a higher conditional expected log excess return on her wealth portfolio as this volatility increases. Moreover, consistent with intuition, expression (92) implies that the conditional expected log excess return on arbitrary asset i is increasing in the conditional covariance of its log excess return with the log excess return on the agent's wealth portfolio. Alternately, if $\psi < 1$, then a preference for early resolution of uncertainty (i.e., $\gamma > \frac{1}{\psi}$) implies that the agent likes the volatility associated with the log excess return on her wealth portfolio and so requires a lower conditional expected log excess return on her wealth portfolio as this volatility increases. Moreover, consistent with intuition, expression (92) implies that the conditional expected log excess return on arbitrary asset i is decreasing in the conditional covariance of its log excess return with the log excess return on the agent's wealth portfolio.

It is now possible to obtain an expression for $r_{f,t+1}$ in terms of utility parameters, variances, covariances and $\mu_{g,t}$. First, it is possible to express $r_{f,t+1}$ in terms of utility parameters, variances, covariances, $\mu_{g,t}$ and $\mathrm{E}_t[r_{W,t+1} - r_{f,t+1}]$. Using (90), we know

$$\begin{aligned} r_{f,t+1} &= -\mathrm{E}_t[m_{t+1}] - \tfrac{1}{2}\sigma_t^2[m_{t+1}] \\ &= -\theta\ln\delta + \frac{\theta}{\psi}\mu_{g,t} + (1-\theta)\mu_{W,t} \\ &\quad - \frac{1}{2}\left(\frac{\theta^2}{\psi^2}\sigma_{g,t}^2 + (\theta-1)^2\sigma_{W,t}^2 - 2\frac{\theta(\theta-1)}{\psi}\sigma_{gW,t}^2\right). \end{aligned}$$

Subtract $(1-\theta)r_{f,t+1}$ from both sides and divide by θ to obtain

$$r_{f,t+1} = -\ln\delta + \frac{1}{\psi}\mu_{g,t} + \frac{1-\theta}{\theta}\,\mathrm{E}_t[r_{W,t+1} - r_{f,t+1}] \\ -\frac{1}{2}\frac{\theta}{\psi^2}\sigma^2_{g,t} - \frac{1}{2}\frac{(\theta-1)^2}{\theta}\sigma^2_{W,t} + \frac{(\theta-1)}{\psi}\sigma^2_{gW,t}.$$

Now substitute in (92) to obtain

$$\begin{aligned} r_{f,t+1} &= -\ln\delta + \frac{1}{\psi}\mu_{g,t} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma^2_{g,t} + \left(\frac{(1-\theta)(\frac{1}{2}-\theta)}{\theta} - \frac{1}{2}\frac{(\theta-1)^2}{\theta}\right)\sigma^2_{W,t} \\ &\quad + \left(\frac{1-\theta}{\theta}\frac{\theta}{\psi} + \frac{\theta-1}{\psi}\right)\sigma_{gW,t} \\ &= -\ln\delta + \frac{1}{\psi}\mu_{g,t} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma^2_{g,t} + \frac{1}{2}\theta(1+2\theta^2-3\theta-\theta^2-1+2\theta)\sigma^2_{W,t} \\ &= -\ln\delta + \frac{1}{\psi}\mu_{g,t} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma^2_{g,t} + \frac{1}{2}(\theta-1)\sigma^2_{W,t}. \end{aligned}$$

Equity premium and riskfree rate puzzles: CRRA versus Epstein-Zin. We compare how $\ln(\mathrm{E}_t\, R_{W,t+1}/R_{f,t+1})$ and $\ln(R_{f,t+1})$ change going from CRRA preferences with $\gamma = 10$ to Epstein–Zin preferences with $\gamma = 10$ and $\psi = 1.5$. Note that

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}} = \frac{1-10}{1-\frac{1}{1.5}} = \frac{1.5(1-10)}{0.5} = -27.$$

for Epstein–Zin preferences. This comparison shows how Epstein-Zin preferences can help with the equity premium puzzle and the risk-free rate puzzle.

First, consider $\ln(\mathrm{E}_t[R_{W,t+1}]/R_{f,t+1})$:

$$\begin{aligned} \text{E–Z:} \qquad \mathrm{E}_t[r_{W,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma^2_{W,t} &= \frac{\theta}{\psi}\sigma_{gW,t} + (1-\theta)\sigma^2_{W,t} \\ &= \frac{-27}{1.5}\sigma_{gW,t} + 28\sigma^2_{W,t} \\ &= -18\sigma_{gW,t} + 28\sigma^2_{W,t} \\ \text{CRRA:} \qquad \mathrm{E}_t[r_{W,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma^2_{W,t} &= 10\sigma_{gW,t}. \end{aligned}$$

Recall that the equity premium puzzle says that the equity Sharpe ratio is too high using CRRA preferences and aggregate consumption data moments. Since $\sigma_{W,t} \gg \sigma_{gW}$, it follows that

$$\mathrm{E}_t[r_{W,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma^2_{W,t}$$

is much higher for Epstein–Zin than power utility, which means Epstein–Zin preferences can help explain the equity premium puzzle.

Second, consider $\ln(R_{f,t+1})$:

$$\begin{aligned}
\text{E–Z:} \qquad r_{f,t+1} &= -\ln\delta + \frac{1}{\psi}\mu_{g,t} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_{g,t}^2 + \frac{1}{2}(\theta-1)\sigma_{W,t}^2 \\
&= -\ln\delta + \frac{2}{3}\mu_{g,t} + \frac{1}{2}\frac{27\times 4}{3\times 3}\sigma_{g,t}^2 - \frac{1}{2}28\sigma_{W,t}^2 \\
&= -\ln\delta + \frac{2}{3}\mu_{g,t} + 6\sigma_{g,t}^2 - 14\sigma_{W,t}^2 \\
\text{CRRA:} \qquad r_{f,t+1} &= -\ln\delta + 10\mu_{g,t} - 50\sigma_{g,t}^2.
\end{aligned}$$

Recall that the risk free rate puzzle says the risk free rate is too low using CRRA preferences and aggregate consumption data moments.. Since $\frac{2}{3}\mu_{g,t} \ll 10\mu_{g,t}$ and $\sigma_{W,t}^2 \gg \sigma_{g,t}^2$, it follows that $r_{f,t+1}$ is lower for Epstein–Zin than power preferences, which means Epstein–Zin preferences can simultaneously help explain the risk free rate puzzle.

13.4. Long run risk model

Model setup. Consider an economy with a representative agent who has Epstein–Zin preferences, and define g_{t+1} to be log consumption growth and $g_{i,t+1}$ to be the log dividend growth of asset i. Suppose

$$\begin{aligned}
g_{t+1} &= \mu + x_t + \sigma\eta_{t+1}, \\
g_{i,t+1} &= \mu_i + \phi_i x_t + \varphi_i\sigma u_{t+1}, \\
x_{t+1} &= \rho x_t + \varphi\sigma e_{t+1},
\end{aligned}$$

where η_{t+1}, u_{t+1} and e_{t+1} are i.i.d. $N(0, I_3)$, I_n is a $n \times n$ identity matrix, and $0 < \rho < 1$. Note that $r_{W,t+1}$ is the return on the total wealth portfolio whose dividend process equals the aggregate consumption process C_{t+1}.

Approximate expression for $r_{W,t+1}$. Let P_t be the price of the aggregate consumption series and define $p_{t+1} = \ln(P_{t+1})$, $c_{t+1} = \ln(C_{t+1})$ and $z_t = \ln(P_t/C_t)$. We can use a log linearization of $c_{t+1} - p_{t+1}$ around its unconditional mean $\bar{c} - \bar{p}$ to derive the approximate relation

$$r_{W,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}, \tag{93}$$

where κ_0 and κ_1 are linearization constants.

To show this, first note that

$$f(x) \approx f(\bar{x}) + f'(\bar{x})(x - \bar{x}),$$

and so taking $f(x) = \ln(1 + \exp x)$, it follows that

$$\ln(1 + \exp x) \approx \ln(1 + \exp(\bar{x})) + \frac{\exp\bar{x}}{1 + \exp\bar{x}}(x - \bar{x}). \tag{94}$$