The Influence of Changes in Asset Volatilities and Correlations on Minimum Variance Portfolio Risk

Abstract

This paper makes a number of contributions to the understanding of minimum variance portfolio (MVP) risk. First, it presents several results connecting changes in MVP risk to changes in portfolio asset volatilities and correlations. Second, it explores the efficacy of three alternative methods of attributing changes in MVP risk to either changes in asset volatility or changes in asset correlations. Lastly, Dow Jones stocks are employed to empirically measure the relative importance of changes these two influences to the dynamics of MVP risk during the period 2001 to 2010 with particular emphasis on the financial crisis years 2008 and 2009.

Key words: Diversification, GFC, Minimum Variance Portfolio, Volatility, Correlation.

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1. Introduction

Holding a diverse range of assets is the most basic, and possibly the most efficacious, strategy to reduce investment risk. While there are diversification benefits for all portfolios constructed from non-perfectly positively related assets, the focus of this note is the diversification/risk characteristics of the so-called global minimum variance portfolio (MVP). In this regard, the study follows Chopra and Ziemba (1993), Ledoit and Wolf (2003), Jagganathan and Ma (2003), Clarke, de Silva and Thorley (2006) and Kempf and Memmel (2006) who also used the MVP as an object of investigation.

By definition, the MVP is the portfolio of choice for any investor whose overriding concern is the avoidance of return risk. Financial crises are invariably characterised by decreases in asset returns and increases in asset return volatility. Any increase in general asset return volatility will increase portfolio risk[1]. However, an MVP investor ought to be able to take comfort in the knowledge that asset diversification will minimise the effect of asset volatility increases will have on portfolio risk during periods of market turmoil.

It is not only asset volatilities that increase during financial crises. It is generally accepted that pair-wise asset return correlations also typically increase during financial crises.[2] Theoretical support for the proposition that asset correlations increase during times of high return volatility is provided by the capital asset pricing model (see Appendix A). Asset correlations determine portfolio diversification gains. Any increase in inter asset correlation reduces diversification gains and thus increases portfolio risk. Increases in asset return correlations during periods of financial crisis may reduce the benefits of diversification at times when investors most need these benefits.

The paper makes a number of contributions. It documents the theoretical influence of asset return volatility and asset correlations have on MVP risk. The empirical section of the paper examines dynamics of MVP risk and changing benefits offered by MVP diversification over the period of the study. Of particular interest is the extent of impairment of asset diversification benefits during the period of the global financial crisis 2008-2009.

In order to quantify the relative importance of asset volatilities and correlations in determining MVP risk, this study proposes and employs three alternative methods for attributing changes in MVP risk to either volatility changes or correlation changes in the underlying assets. These three methods are applied to Dow Jones stocks, MVP risk over the sample period 2001 to 2010.

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[1] Volatility refers to annualised standard deviation of asset returns. This paper uses risk and volatility as synonymous terms.

[2] For example, Ang and Chen (2001) found correlation between stocks and the market increased during market downturns.

2. MVP Characteristics

The MVP is the portfolio, formed from p assets, with the lowest possible portfolio return variance for a given portfolio covariance of matrix, $\mathbf{\Omega}$. Let $\mathbf{w} = (w_1, \ldots w_p)'$ represent a set of portfolio weights that sum to unity, ie $\boldsymbol{\iota}'\mathbf{w} = 1$, where $\boldsymbol{\iota} = (1, \ldots 1)'$. The portfolio variance associated with a weights vector $\mathbf{w}$, σ^2 is given by the following quadratic equation:

$$\sigma^2 = \mathbf{w}' \, \mathbf{\Omega} \, \mathbf{w} \tag{1}$$

The short-sales allowed MVP asset weights vector, $\mathbf{w_m}$, found by minimising portfolio variance subject to the constraint the weights sum to zero is:

$$\mathbf{w_m} = \mathbf{\Omega}^{-1} \boldsymbol{\iota} / (\boldsymbol{\iota}' \, \mathbf{\Omega}^{-1} \boldsymbol{\iota}) \tag{2}$$

The no-short-sales MVP is the solution to the more restricted problem where the asset weights are restricted to having non-negative values. The no-short-sales MVP is a specific example of a solution to a more general quadratic programming problem where assets that would otherwise take a negative weight in the short-sales-allowed MVP are "excluded" from the portfolio by having their weights set to zero.[3] The number of assets "included" in the no-short-sales MVP depends upon the value of elements in the covariance matrix $\mathbf{\Omega}$. (see Best and Grauer (1992)).

Any no-short-sales risk/return frontier is actually a series of piece-wise, short-sales-allowed risk/return frontiers. Discontinuity in any no-short-sales risk/return frontier occurs at the points where assets are added to, or dropped from, the no-short-sales portfolio. The results presented below implicitly assume short-sales-allowed continuity in a portfolio formed from an asset set determined by the short-sales-not-allowed optimising technique.[4]

Substituting the solution for the no-short-sales, MVP, asset weights vector, $\mathbf{w}_m$, (equation (2)) into the portfolio variance equation (1) yields the MVP risk/volatility, σ_m:

$$\sigma_m = 1 / \sqrt{\boldsymbol{\iota}' \mathbf{\Omega}^{-1} \boldsymbol{\iota}} \tag{3}$$

In order to disentangle the effects that changing asset volatilities and changing asset correlations have on MVP risk, σ_m, it is instructive to decompose the covariance matrix, $\mathbf{\Omega}$, into these two elements. As each covariance element $\sigma_{i,j} = \sigma_i \, \sigma_j \, r_{i,j}$, where $r_{i,j}$ is the correlation between the return on asset i and the return on asset j, the covariance matrix can be decomposed into a quadratic form:

$$\mathbf{\Omega} = \mathbf{SRS} \tag{4}$$

[3] This study employs a version of the simplex method to solve the quadratic programming problem.
[4] Henceforth it is assumed that n assets ($n \leq p$) have non-zero weights in the MVP.

where **S** is a diagonal matrix, with each diagonal element consisting of an asset return volatility and **R** is a correlation matrix. The structure of S and R are as follows:

$$\mathbf{S} = \begin{bmatrix} \sigma_1 & . \ . & 0 \\ . & & . \\ & \sigma_i & \\ . & & . \\ 0 & . \ . & \sigma_n \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} 1 & r_{1,2} & . & r_{1,n} \\ r_{2,1} & & & . \\ & 1 & & \\ . & & & r_{n-1,n} \\ r_{n,1} & . & r_{n,n-1} & 1 \end{bmatrix}$$

It is apparent from equation (4) that there are there are n (n+1)/2 distinct elements that determine the composition of $\mathbf{\Omega}$ and thus the structure of the MVP portfolio and its associated risk measure, σ_m. These elements can be separated into two vectors, an n element vector of asset return standard deviations, $\mathbf{s}= (\sigma_1, \sigma_2, \ \ldots \ \sigma_n)'$, and a n (n-1)/2 element vector of distinct pair-wise correlations, $\mathbf{r} = (r_{2,1}, r_{3,1} \ \ldots \ r_{n,1}, r_{3,2} \ \ldots. \ r_{n-1,n})'$ that reside beneath the diagonal elements of **R**.

This study investigates the causes of changes in MVP risk. Central to this purpose is an exploration of the affect that changes in asset volatilities and correlations have on MVP risk. Insight into the relationship between asset volatilities and asset correlations and MVP risk is provided by partial derivatives of MVP risk with respect to asset volatilities ($\delta\sigma_m/\delta\mathbf{s}$) and with respect to asset correlations ($\delta\sigma_m/\delta\mathbf{r}$). Derivations of these partial derivatives are set out in Appendix B.

An obvious cause of change in MVP volatility is variation in the value of volatility of individual assets. The sensitivity of the MVP risk to changes in individual asset return volatility has the following form:

$$\frac{\delta\sigma_m}{\delta\mathbf{s}} = \sigma_m \mathbf{w}_\mathbf{m}' \mathbf{S}^{-1} \tag{5}$$

or

$$\frac{\delta\sigma_m}{\delta\sigma_\mathbf{i}} = w_{m,i}\sigma_m/\sigma_i \qquad i=1,n \tag{6}$$

Note $0 \le \delta\sigma_m / \delta\sigma_i \le 1$ as by construction $0 \le w_{m,i} \le 1$ and $\sigma_m \le \sigma_i$ for all assets. This result confirms the intuitive understanding of the influence of individual asset volatilities have on MVP risk. That is, any increase in the volatility of any individual asset in the MVP results in an increase in MVP risk. However, the increase in MVP risk is smaller than the increase in the volatility of the asset.

Changes in asset return correlations provide a second source of change in MVP risk. The sensitivity of MVP risk to any change in the value of the individual asset return correlations may be examined by differentiating MVP risk with respect to the correlation vector, **r**. This produces the following equation:

$$\frac{\delta\sigma_m}{\delta\mathbf{r}} = \frac{1}{2\sigma_m}(\mathbf{w}_m'\otimes\mathbf{w}_m')(\mathbf{S}\otimes\mathbf{S})\mathbf{D} \tag{7}$$

or

$$\frac{\delta\sigma_m}{\delta r_{i,j}} = w_i\sigma_i w_j\sigma_j/\sigma_m \qquad \text{i=2,n} \quad \text{j=i+1,n-1} \tag{8}$$

Where **D** a duplicator matrix that selects for the infradiagonal elements of **R.** (See Appendix B for more detail). Note, that as all elements of the right hand side of (8) are non-negative for any no-short-sales MVP, $\delta\sigma_m / \delta r_{i,j} \geq 0$. The non-negativity of these individual elements ensures that MVP volatility is positively related to each pair-wise correlation. An increase in the correlation of the returns on any asset pair increases the risk associated with the minimum risk portfolio.

Equations (5) to (8) document the effect that a change in any single asset return volatility, σ_i or in any single asset pair correlation, $r_{i,j}$ has on MVP risk. The number of these micro-level, influencing pathways provides an analytical challenge. One response to this challenge of the multiplicity of influences is to explore changes in MVP risk, and the causes thereof, at a more macro level than the individual asset return volatilities and correlations. This is achieved using the approximation (9) below.

Assume that within the set of investment assets there is an "average" volatility, $\bar{\sigma}$ and an average pair-wise return correlation, $\bar{r}$. If additionally it is assumed that the MVP weights vector, $\mathbf{w}_m$, is relatively well dispersed, then the following approximation holds:[5]

$$\sigma_m \approx \bar{\sigma}\sqrt{\bar{r}} \tag{9}$$

Here, MVP risk is approximately equal to the average asset volatility multiplied by the square root of the average asset pair correlations. This approximation represents a simplification whereby the influences MVP risk are reduced from n (n+1)/2 to 2, where n is the number of assets included in the MVP. However, the analytical attractiveness of approximation needs to be weighed against the inherent error contained within. For example, the MVP risk, σ_m, for the whole period 2001 to 2010 is shown in Table 1 below as 15.5%pa. Using values, in Table 2, for the average stock risk, $\bar{\sigma}$, and average correlation, approximation is evaluated as:

$$\begin{aligned}\sigma_m &\approx \bar{\sigma}\sqrt{\bar{r}} = 0.221 * \sqrt{0.3988}\\ &= 14.0\%\text{pa}\end{aligned}$$

[5] See appendix C for a derivation that follows a a similar to method to that of Elton *et al* (2007) p58.

Thus the approximation, in this example, has an error of 1.5%pa.[6]

3. Data

This study uses daily return data, for the periods from 2000 to 2010, on 26 Dow Jones Industrial Average index (DJIA) stocks as the basis for its empirical analysis. The 26 stocks selected for analysis were those stocks, that were part of the (DJIA) in 2010, and that provided daily returns to the CRSP database over the period 2000 to 2010.[7] The use of DJIA stocks follows Engle and Sheppard's (2001) study of conditional correlation that employs DJIA stock returns, but over a shorter period.

Dow Jones stocks were chosen for analysis for several reasons. First, these stocks are important international assets being the traded shares of the US's largest and most liquid companies. Second, the 26 stocks provided a sufficient number of assets to ensure representative results for an equity portfolio study, without providing an undue computation and comprehension burden that would accompany the use of stocks from a broader share price index.

Statistics on the DJIA stocks are set out in Table 1 for each calendar year for the period 2001 to 2010. Table 1 show that both average volatility and correlation was greatest during the financial crisis year of 2008. Table 1 also shows that there was also considerable fluctuation in stock volatility and correlation during the dot com boom and bust in the early years of the first decade of the 21st century.

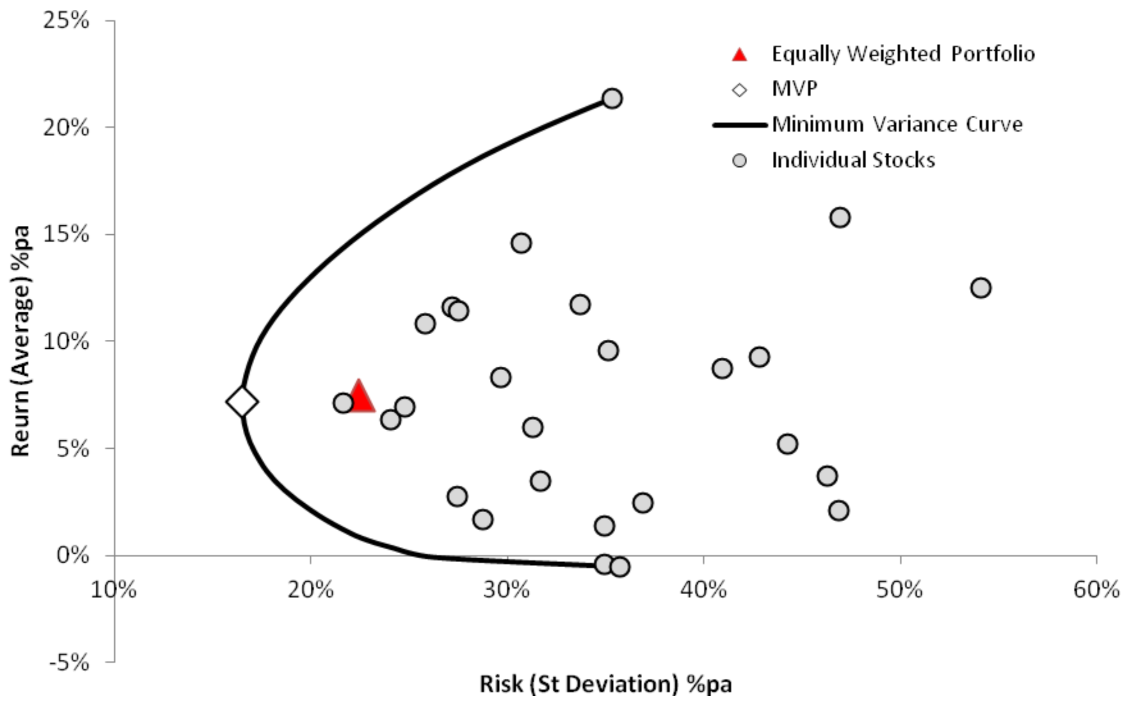
Table 1 records MVP statistics for each year 2001 to 2010. The pattern of MVP volatility follows that of the average volatility being high during the early years of the decade and during GFC crisis year of 2008. It can be seen that the number of stocks included in the MVP was greatest during the relatively tranquil years of the mid part of the decade.

[6] The approximation, which applies only if $\overline{r}>0$, is better, the larger the number of assets in the MVP and, the closer the MVP is to an equally-weighted portfolio.
[7] The ticker symbols of the 26 stocks included in the study are: AA, AXP, BA, BAC, CAT, CSCO, DD, DIS, GE, HD, HWP, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, PFE, PG, T, UTX, WMT and XOM.

Table 1: DJIA Stock Statistics					
	Average Volatility.* (%pa)	Average Correlation*	Stocks Included in MVP	MVP Volatility (%pa)	MVP Return (%pa)
2001	28.0%	0.1509	12	13.8%	-0.6%
2002	29.2%	0.2942	10	19.2%	-2.2%
2003	16.6%	0.4071	7	12.2%	20.3%
2004	15.6%	0.2663	13	9.0%	14.2%
2005	14.3%	0.2962	13	8.7%	1.9%
2006	13.6%	0.2868	15	8.1%	20.1%
2007	13.6%	0.4053	7	10.1%	11.8%
2008	32.0%	0.6733	6	27.5%	-1.6%
2009	20.6%	0.3656	6	14.9%	10.8%
2010	13.6%	0.4878	5	10.6%	7.1%
2001-2010	22.1%	0.3988	8	15.5%	7.2%
* The average returns, standard deviations and correlations are weighted by MPV asset weights					

Figure 1 depicts the no-short-sales minimum variance curve for the period 2001 to 2010. The individual stock risk/return coordinates and the equally weighted portfolio are also displayed in Figure 1. The difference between the average volatility figure of 22.1%pa and the MVP volatility number of 15.5%pa can be interpreted as the diversification gain that arises from DJIA stocks not being perfectly correlated with each other.

Figure 1: DJIA Stocks Risk and Return



4. Attribution of Changes in MVP risk

Changes in MVP risk from period to period, arise because of changes in the elements of the covariance matrix, Ω. The elements of Ω are themselves functions of a vector of asset return volatilities, **s**, and a vector of asset return correlation pairs, **r**. Thus changes in MVP risk from period to period, arise may be attributed to either (1) changes in asset volatility (2) changes in asset correlations or (3) by some residual term caused non-linearity/interaction between (1) and (2). Thus change in MVP risk from period to period, $\Delta\sigma_m$, can be expressed as:

$$\Delta\sigma_m = \Delta\sigma_m^{vol} + \Delta\sigma_m^{corr} + e \tag{10}$$

Where, $\Delta\sigma_m^{vol}$ is the change in MVP risk due to changes in asset volatilities and $\Delta\sigma_m^{corr}$ is the change in MVP risk due to changes in correlation numbers. The residual term, e, is an error term. We explore three methods of accounting for changes in MVP risk that can be directly attributed to changes in either asset volatility factors or asset correlation factors.

4.1 Macro-Level Method of Attribution

Approximation (9) can be manipulated to produce an equation linking changes in MVP risk to changes in average volatility and average correlation.

$$\Delta\sigma_m = \sqrt{\bar{r}}\,\Delta\bar{\sigma} + \left(\frac{\bar{\sigma}}{2\sqrt{\bar{r}}}\right)\Delta\bar{r} + e \tag{11}$$

Where $\Delta\sigma_m$ is the change in MVP risk from period t-1 to period t, $\bar{r}$ is the weighted average of stock correlations at period t-1, $\bar{\sigma}$ is the weighted average of stock volatilities at period t-1 and e is an interaction/error term.

4.2 Micro-Level Method of Attribution

Equation (5) quantifies the effect of changes in individual volatilities on MVP risk. Similarly, Equation (7) summarises the effect that changes in individual correlation pairs have on MVP risk. A micro-level attribution equation can be expressed as:

$$\Delta\sigma_m = \frac{\delta\sigma_m}{\delta\mathbf{s}}\Delta\mathbf{s} + \frac{\delta\sigma_m}{\delta\mathbf{r}}\Delta\mathbf{r} + e \tag{12}$$

Where the numerical value of the elements of $\delta\sigma_m/\delta\mathbf{s}$ and $\delta\sigma_m/\delta\mathbf{r}$ are specified by equation (5) and equation (7) respectively. Equation (12) justifies the label micro level as there are 26 elements in $\Delta\mathbf{s}$ and 325 elements in $\mathbf{\Delta r}$.

4.3 Isolating Changes in Volatilities and Changes in Correlations in Ω

This method of attribution relies on artificially isolating the source of changes in covariance matrix $\boldsymbol{\Omega}$. The method employs two modified versions of the covariance matrix. $\boldsymbol{\Omega}^{vol}$ is a covariance matrix constructed using correlation values from the previous period, t-1, but with asset volatility values from this period, t. Similarly, $\mathbf{\Omega}^{corr}$ is a covariance matrix constructed using last period's volatilities and this period's correlation numbers. In terms of equation (10), $\Delta\sigma_m^{vol}$ is calculated as the difference between last period's MVP volatility and the MVP volatility using computed using $\mathbf{\Omega}^{vol}$ where asset correlations remain the same as the previous period and only asset volatilities have been updated to this period. Similarly $\Delta\sigma_m^{corr}$ is calculated using a covariance matrix where only the asset correlations have been updated.

5. Results

Table 2 displays the results of the three methods of attributing changes in MVP risk to either volatility or correlation changes. Panel A of Table 2 shows results pertaining to annual changes for the period 2001 to 2010. Panel B of Table 2 set out statistics relating to the most volatile period July 2008 to June 2009.

The third, fourth and fifth columns of Table 2 record changes in the MVP risk, changes in average stock return volatility and changes in average correlation between stock returns respectively. Figures in Panel A of Table 2 show that the calendar year 2008 was a most volatile period with MVP risk rising 17.4%pa, average volatility increasing by 18.3%pa and correlation increasing by 0.27. Moreover, the average volatility and correlation figures for 2008 and 2009 show the statistics moving strongly in the same direction; up in 2008 and down in 2009.

The two monthly change analysis in Panel B of Table 2 takes a closer look at the dynamics MVP volatility change during the 2008–2009 period. The figures in Panel B provide further evidence in support of the view that asset volatility and cross asset correlation move in tandem.

Table 2 includes two summary statistics, standard deviation and average absolute change, to measure the relative influence of volatility changes and correlation changes on changes in MVP risk. The summary statistics indicate that all three methods of attributing the cause of changes in MVP risk lead to similar conclusions. Namely:

1. the interaction/residual error term is substantial,
2. correlation changes are a significant factor in producing changes in MVP risk volatility however,
3. volatility changes are more important than correlation changes in the dynamics of MVP risk.

Table 2: Attributing Changes in MVP Risk

Panel A *Year on Year Changes 2001 to 2010*

Period	$\Delta\sigma_m$	$\Delta\bar{\sigma}$	$\Delta\bar{r}$	Macro Attribution (4.1)			Micro Attribution (4.2)			Substitution Attribution(4.3)		
				Vol	Correl	Error	Vol	Correl	Error	Vol	Correl	Error
2001	-3.1%	-14.2%	0.06	-4.2%	4.5%	-3.4%	-3.5%	5.5%	-5.0%	-4.4%	2.6%	-1.2%
2002	5.4%	1.3%	0.14	0.5%	5.2%	-0.2%	2.6%	5.8%	-3.0%	1.6%	3.8%	0.0%
2003	-7.0%	-12.6%	0.11	-6.9%	3.0%	-3.2%	-5.7%	0.6%	-2.0%	-7.1%	-0.2%	0.3%
2004	-3.2%	-1.0%	-0.14	-0.6%	-1.8%	-0.7%	-0.2%	-0.5%	-2.5%	-2.1%	-1.4%	0.4%
2005	-0.3%	-1.3%	0.03	-0.7%	0.5%	-0.1%	0.4%	0.5%	-1.3%	-1.0%	0.1%	0.6%
2006	-0.6%	-0.7%	-0.01	-0.4%	-0.1%	0.0%	0.0%	0.2%	-0.8%	-0.6%	-0.4%	0.4%
2007	1.9%	0.1%	0.12	0.0%	1.5%	0.4%	1.6%	1.9%	-1.5%	0.6%	0.8%	0.5%
2008	17.4%	18.3%	0.27	11.7%	2.9%	2.9%	14.2%	2.0%	1.2%	12.4%	1.1%	3.9%
2009	-12.6%	-11.4%	-0.31	-9.3%	-6.0%	2.7%	-9.2%	-3.6%	0.2%	-10.4%	-4.4%	2.2%
2010	-4.2%	-7.0%	0.12	-4.2%	2.1%	-2.1%	-4.7%	1.4%	-0.9%	-4.9%	1.1%	-0.4%
St. Dev.	7.6%	8.9%	0.15	5.4%	3.1%	2.0%	6.0%	2.6%	1.6%	5.8%	2.1%	1.4%
Avg. Abs	5.6%	6.8%	0.13	3.9%	2.8%	1.6%	4.2%	2.2%	1.8%	4.5%	1.6%	1.0%

Panel B *Two Month on Two Month Changes July 2008 to June 2009*

Period	$\Delta\sigma_m$	$\Delta\bar{\sigma}$	$\Delta\bar{r}$	Macro Attribution			Micro Attribution			Substitution Attribution		
				Vol	Correl	Error	Vol	Correl	Error	Vol	Correl	Error
Jul-Aug08	-0.2%	1.4%	-0.24	0.8%	-3.7%	2.8%	1.5%	-0.4%	-1.3%	1.2%	-1.3%	-0.1%
Sep-Oct08	14.7%	11.1%	0.60	2.9%	21.4%	-9.5%	10.4%	5.7%	-1.4%	9.0%	2.3%	3.4%
Nov-Dec08	20.4%	22.0%	0.10	18.0%	1.8%	0.7%	21.8%	1.7%	-3.1%	17.0%	1.0%	2.4%
Jan-Feb09	-25.4%	-21.9%	-0.38	-19.2%	-11.1%	4.9%	-17.9%	-7.2%	-0.2%	-22.1%	-10.9%	7.6%
Mar-Apr09	-3.5%	-5.4%	-0.01	-3.4%	-0.2%	0.0%	-1.2%	0.8%	-3.1%	-3.4%	-1.1%	0.9%
May-Jun09	-4.8%	-5.5%	-0.06	-3.4%	-1.2%	-0.2%	-2.8%	0.4%	-2.4%	-4.8%	-0.5%	0.6%
St. Dev.	14.8%	13.8%	0.31	10.9%	9.9%	4.5%	12.2%	3.8%	1.0%	12.2%	4.3%	2.6%
Avg. Abs	11.5%	11.2%	0.23	7.9%	6.6%	3.0%	9.3%	2.7%	1.9%	9.6%	2.8%	2.5%

$\Delta\sigma_m$ is the change in MVP risk, $\Delta\bar{\sigma}$ is the change in average return volatility (standard deviation), $\Delta\bar{r}$ is the change in average correlation and Vol is volatility, Corr is correlation. St.Dev. is standard deviation and Abs. Avg. is the average of absolute changes.

Of the three attribution methods, the micro-level method 4.2 is the most appealing in terms of academic rigour. However, method 4.3 resulted in the smallest error as measured by either the variance of the error or the average absolute value of the error. Method 4.1, the macro level, average values method, was the least precise attribution as judged by the size of the error terms.

Using the relative size of the volatility effect compared to the correlation effect, as determined by methods 4.2 and 4.3, it is reasonable to conclude that the correlation effect on MVP risk is less than half the volatility effect. Surprisingly, while changes to correlations exacerbated volatility induced changes in MVP risk, during the two periods of greatest change in MVP risk, namely the year 2008 and the period Nov. 2008 to Dec. 2008, the correlation effect was dwarfed by the volatility effect.

6. Summary

The study uses daily observations on DJIA traded shares to investigate the influence that asset return volatilities and asset return correlations have on the risk/volatility of the minimum variance portfolio. One of the contributions of this study was to develop several approaches to disentangle the effects that volatilities and correlations have on MVP risk.

The study found that asset volatilities and asset correlations were far from stationary, and that significant movements in these variables took place over the period of the study, the decade 2001 to 2010. These movements were particularly pronounced during the share-market crisis years of 2008-2009.

The investigation found that changes in asset volatilities and asset correlations are significant determinants of change in MVP risk. While changes in asset correlations exacerbated, rather than ameliorated, changes in MVP risk due to changes in stock volatilities, the effect was less. On average the effect of changes in stock volatilities is of an order of two to three times that of changes in stock correlations during the period of study.

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7. Appendix

7.1 Appendix A

Under the single index model, the correlation between the returns on two assets, i and j can be expressed as:

$$r_{i,j} = \frac{\beta_i \beta_j \sigma^2_{mkt}}{\sigma_i \sigma_j} = \frac{\beta_i \beta_j \sigma^2_{mkt}}{\sqrt{(\beta_i^2 \sigma^2_{mkt} + \sigma^2_{e,i})(\beta_j^2 \sigma^2_{mkt} + \sigma^2_{e,j})}} \tag{A1}$$

Where σ^2_{mkt} is the variance of the market, β_i and β_j are stock betas and σ_i and σ_j are stock total risk. Assuming that the idiosyncratic risk factors, $\sigma^2_{e,i}$ and $\sigma^2_{e,j}$ are fixed, its is clear from (A1) that in the limit as σ^2_{mkt} becomes large, $r_{i,j}$ asymptotes to either +1 (β_1 and β_j have the same sign) or -1 (β_1 and β_j have the opposite sign). The derivative of $r_{i,j}$ w.r.t, σ_{mkt} is given by:

$$\frac{dr_{i,j}}{d\sigma_{mkt}} = \frac{\beta_i \beta_j}{\sigma_i \sigma_j}\left[1 - \frac{\sigma^2_{mkt}(\beta_i^2 \sigma_j^2 + \beta_j^2 \sigma_i^2)}{\sigma_i \sigma_j}\right] \tag{A2}$$

If β_1 and β_j have the same sign the derivative is positive otherwise the reverse holds. As to be expected, the value of the derivative (A2) approaches zero as σ_{mkt} becomes large.

7.2 Appendix B

Let us examine the sensitivity of MVP risk, σ_m, to changes in asset volatilities. From equation (6) it is apparent that

$$\begin{aligned} \frac{\delta \sigma_m}{\delta \mathbf{s}} &= \left(\frac{\delta \sigma_m}{\delta \sigma_1}, \,\dots\dots\dots, \frac{\delta \sigma_m}{\delta \sigma_n} \right) \\ &= \frac{\delta \sigma_m}{\delta \sigma_m^2} \frac{\delta \sigma_m^2}{\delta d} . \frac{\delta d}{\delta \mathbf{s}^{-1}} . \frac{\delta \mathbf{s}^{-1}}{\delta \mathbf{s}} \end{aligned} \tag{B1}$$

Where σ^2=1/d, $d = \boldsymbol{\iota}' \, \boldsymbol{\Omega}^{-1} \boldsymbol{\iota}$ and $\mathbf{s}^{-1}$= (1/$\mathbf{s}_1$, … , 1/$\mathbf{s}_n$)

Now,

$$\frac{\delta \sigma_m}{\delta \sigma_m^2} = \frac{1}{2\sigma_m} \tag{B2}$$

$$\frac{\delta \sigma_m^2}{\delta d} = -\frac{1}{d^2} = -\sigma_m^4 \tag{B3}$$

$$d = \boldsymbol{\iota}' \, \boldsymbol{\Omega}^{-1} \boldsymbol{\iota} = \boldsymbol{\iota}' \, (\mathbf{SRS})^{-1} \boldsymbol{\iota} = \boldsymbol{\iota}' \, \mathbf{S}^{-1}\mathbf{R}^{-1}\mathbf{S}^{-1}\boldsymbol{\iota} = (\mathbf{s}^{-1})' \, \mathbf{R}^{-1}\mathbf{s}^{-1} \tag{B4}$$

$$\frac{\delta d}{\delta \mathbf{s}^{-1}} = 2(\mathbf{s}^{-1})'\mathbf{R}^{-1} \tag{B5}$$

Finally:

$$\frac{\delta \mathbf{s}}{\delta \mathbf{s}^{-1}} = -\mathbf{s}^{-2} \tag{B6}$$

Thus

$$\begin{aligned} \frac{\delta\sigma_m}{\delta s} &= \frac{1}{2\sigma_m}(-\sigma_m^4)(2(\mathbf{s}^{-1})'\mathbf{R}^{-1})(-\mathbf{S}^{-2}) \\ &= \sigma_m^3 \boldsymbol{\iota}'(\mathbf{S}^{-1}\mathbf{R}^{-1}\mathbf{S}^{-1})\mathbf{S}^{-1} \\ &= \sigma_m^3 \boldsymbol{\iota}'\boldsymbol{\Omega}^{-1}\mathbf{S}^{-1} \\ &= \sigma_m w_m' \mathbf{S}^{-1} \\ &= \sigma_m (w_1/\sigma_1, \ \ldots\ldots\ldots \ , w_n/\sigma_n) \end{aligned} \tag{B7}$$

as $\mathbf{s}^{-1} = \boldsymbol{\iota}'\mathbf{S}^{-1}$, $\mathbf{w_m} = \boldsymbol{\iota}'\boldsymbol{\Omega}^{-1}/\mathrm{d}$, $\sigma_m^2 = 1/\mathrm{d}$ and $\mathbf{S}^{-1}\mathbf{R}^{-1}\mathbf{S}^{-1} = \boldsymbol{\Omega}^{-1}$

Let us now concentrate on the influence of asset correlation on MVP risk. The vector, **r** is composed of the n(n-1)/2 correlation coefficients that populate **R** beneath the principle matrix diagonal.

$$\mathbf{r} = \mathrm{vecl}(\mathbf{R}) \tag{B8}$$

Where vecl is an operator that stacks the elements of a matrix below the principle diagonal. That is **r** = ($r_{2,1}$, .$r_{3,1}$, ….$r_{n,1}$, $r_{3,2}$, ….., $r_{n,n-1}$)’. For each n-asset portfolio there exists a unique “duplication” matrix, D, that maps the elements of **r** into **R** via the following relationship:

$$\mathrm{vec}(\mathbf{R}) = \mathbf{D}\mathbf{r} + \mathrm{vec}(\mathbf{I}_n) \tag{B9}$$

Note that **D** is a n^2*(n(n-1)/2) matrix whose rows consist of zeros and at most one one.[8] Expanding Ω as per equation (4):

$$\begin{aligned} \sigma_m^2 &= \mathbf{w_m}'(\mathbf{SRS})\mathbf{w_m} \\ &= ((\mathbf{w_m S})' \otimes (\mathbf{w_m S})')\mathbf{vec(R)} \\ &= ((\mathbf{w_m S})' \otimes (\mathbf{w_m S})')(\mathbf{D}^* \mathbf{r} + \mathbf{vec}(\mathbf{I_n})) \end{aligned} \tag{B10}$$

The partial derivative of MVP risk with respect to the correlations vector can be expressed as:

$$\begin{aligned} \frac{\delta\sigma_m}{\delta r} &= \left(\frac{\delta\sigma_m}{\delta r_{1,2}}, \ \ldots\ldots\ldots \ , \frac{\delta\sigma_m}{\delta r_{n,n-1}} \right) \\ &= \frac{\delta\sigma_m}{\delta\sigma_m^2} \cdot \frac{\delta\sigma_m^2}{\delta d} \cdot \frac{\delta d}{\delta \mathrm{vec}(\boldsymbol{\Omega}^{-1})} \cdot \frac{\delta \mathrm{vec}(\boldsymbol{\Omega}^{-1})}{\delta \mathrm{vec}(\boldsymbol{\Omega})} \cdot \frac{\delta \mathrm{vec}(\boldsymbol{\Omega})}{\delta \mathrm{vec}(\mathbf{R})} \cdot \frac{\delta \mathrm{vec}(\mathbf{R})}{\delta \mathbf{r}} \end{aligned} \tag{B11}$$

Evaluating the last four terms on the right hand side of (B11).

[8] Further explanation of the properties of duplication matrices is provided by Magnus and Neudekker (1999).

$$\frac{\delta d}{\delta vec(\mathbf{\Omega}^{-1})} = (\iota' \otimes \iota'),\ \frac{\delta vec(\mathbf{\Omega}^{-1})}{\delta vec(\mathbf{\Omega})} = -(\mathbf{\Omega}^{-1} \otimes \mathbf{\Omega}^{-1}),\ \frac{\delta vec(\mathbf{\Omega})}{\delta vec(\mathbf{R})} = (\mathbf{S} \otimes \mathbf{S}),\ \frac{\delta vec(\mathbf{R})}{\delta \mathbf{r}} = \mathbf{D} \qquad \text{(B12)}$$

Thus

$$\begin{aligned} \frac{\delta\sigma_m}{\delta\mathbf{r}} &= \frac{1}{2\sigma_m}\left(-\sigma_m^4\right)(\iota' \otimes \iota')(-\Omega^{-1} \otimes \Omega^{-1})(S \otimes S)D \\ &= \frac{\sigma_m^3}{2}(\iota'\Omega^{-1} \otimes \iota'\Omega^{-1})(S \otimes S)D \end{aligned} \qquad \text{(B13)}$$

Equation (B13) can be simplified by recognising that $\boldsymbol{\iota}'\mathbf{\Omega}^{-1} = \mathbf{w_m} / \sigma_m^2$.
Thus

$$\frac{\delta\sigma_m}{\delta r} = \frac{1}{2\sigma_m}(w_m' \otimes w_m')(S \otimes S)D \qquad \text{(B14)}$$

Inspection of the structure of D* leads to the further simplification of (B17) as follows:

$$S\frac{\delta\sigma_m}{\delta r_{i,j}} = w_{m,i} w_{m,j} \sigma_i \sigma_j / \sigma_m \qquad 1 \le i \le n, i+1 \le j \le n \qquad \text{(B15)}$$

7.3 Appendix C

MVP variance, σ^2_m can be expressed as:

$$\sigma_m^2 = \sum_{i=1}^{n} w_{m,i}^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i \neq j}}^{n} w_{m,i} w_{m,j} r_{i,j} \sigma_i \sigma_j \qquad \text{(C1)}$$

Where $w_{m,i}$ is the weight of the i[th] asset in the MVP. If we accept a concept of the existence of an average asset volatility, $\bar{\sigma}^2$, and an average asset return, $\bar{r}$ and further that we adopt an assumption that MVP weights are equal, that is, $w_{m,i}$ = 1/n where n is number assets in the MVP portfolio, then:

$$\sigma_m^2 = \frac{1}{n}\bar{\sigma}^2 + \frac{n-1}{n}\bar{\sigma}^2\ \bar{r} \qquad \text{(C2)}$$

If n is sufficiently large then

$$\sigma_m^2 \simeq \bar{\sigma}^2\ \bar{r} \quad \text{and thus } \sigma_m \simeq \bar{\sigma}\sqrt{\bar{r}} \qquad \text{(C3)}$$