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Inventories, Fluctuations and Business Cycles

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The paper looks at the role of inventories in U.S. business cycles and fluctuations. It concentrates upon the goods producing sector and constructs a model that features both input and output inventories. A range of shocks are present in the model, including sales, technology and inventory cost shocks. It is found that the presence of inventories does not change the average business cycle characteristics in the U.S. very much. The model is also used to examine whether new techniques for inventory control might have been an important contributing factor to the decline in the volatility of US GDP growth. It is found that these would have had little impact upon the level of volatility.

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Abstract

The paper looks at the role of inventories in U.S. business cycles and fluctuations. It concentrates upon the goods producing sector and constructs a model that features both input and output inventories. A range of shocks are present in the model, including sales, technology and inventory cost shocks. It is found that the presence of inventories does not change the average business cycle characteristics in the U.S. very much. The model is also used to examine whether new techniques for inventory control might have been an important contributing factor to the decline in the volatility of US GDP growth. It is found that these would have had little impact upon the level of volatility.

1. Introduction

It is not uncommon for commentators on the prospects for an economy to draw attention to recent inventory movements. Thus, if there has been a run down in stocks below what is perceived to be normal levels, this is taken as a sign of favorable output prospects in future periods; the reasoning behind this conclusion being that output not only needs to be produced to meet sales, but also to replenish stocks. Early in the history of business cycle research the question arose of whether the presence of inventory holdings by firms was a contributor to the “up and down” movements seen in economies. The classic analyses of this question were by Metzler (1941), (1947), who concluded that “An economy in which businessmen attempt to recoup inventory losses will always undergo cyclical fluctuations..”. His model emphasized the fact that a business would attempt to keep inventories as a proportion of expected sales and so would re-build these if they declined below that target level. Given that sales had to be forecast from their past history, he showed that output would follow a second order difference equation which would have complex roots in many cases. Consequently his model produced a periodic cycle in output and this constituted the foundation of his conclusion. Of course the fact that a periodic cycle can be generated does not mean that it is an important one since the amplitude could be quite small.

Many applications of this methodology were made e.g. Duffy and Lewis (1975). But, after Metzler’s work, inventory research shifted towards deriving optimal rules for stock holdings that balanced the cost of being away from a target level against the cost of the sharp output changes that would be needed if any given level of sales was to be met automatically by rapid output adjustment. The classic

work in this vein is by Holt et al. (1960) and a good summary of the type of model that results is Rowley and Trivedi(1975, ch 2). This strand of research produced optimal decision rules for inventory holdings that effectively rationalized the ad hoc rules that underlay Metzler's models. The hallmark of these models is that there are some exogenous driving forces such as sales and cost shocks and then optimal decisions are made in response to what is known about them. A large body of literature has used models of optimal inventory holdings in empirical work- see Blinder and Maccini (1991) and Ramey and West (1999) for surveys of the literature.

A fundamental problem with Metzler's analysis was that it concentrated upon the possibility of a periodic cycle in output. To explain the difficulties with Metzler's approach consider a series q_t generated as an AR(2) of the form

$$(\Delta q_t - \mu) = \alpha_1(\Delta q_{t-1} - \mu) + \alpha_2(\Delta q_{t-2} - \mu) + \sigma \varepsilon_t, \quad (1.1)$$

where ε_t is *i.i.d*(0, 1). If we ignore the shock ε_t and look at the series q_t^* generated by setting it to zero, then the time between turning points in q_t^* is determined by the magnitude of any complex roots of the polynomial $(1 - \alpha_1 L - \alpha_2 L^2) = 0$, where L is the lag operator. For a series measuring the level of economic activity, turning points in it marked out expansions and contractions, so that turning points in q_t^* determine how long expansions and contractions in q_t^* are. Since q_t^* can be represented by a sinusoidal wave, the duration of time between these turning point is fixed and it seems appropriate to call this a periodic cycle.

In practice we measure the characteristics of cycles using the observed data q_t and not the "latent" data q_t^* - this is the way in which the NBER in the U.S., and the myriad of agencies around the world who follow their approach, measure the business cycle, and is the most common way business cycles are described in textbooks and lectures in macroeconomics. There can be very big differences in the cycle characteristics of q_t^* and those of q_t e.g. Harding and Pagan (2006) show that a model that has roots which imply an average cycle of 22 quarters in q_t^* would have a cycle length of 12 quarters in q_t .¹ Moreover, when one looks at the cycle in this way there is no longer any need for output to follow a second order difference equation with complex roots in order to produce a business cycle.

¹This result does not depend upon the standard deviation σ as the turning points in q_t and $\sigma^{-1}q_t$ are identical, provided $\sigma \neq 0$. Note that any quantities that depend solely upon the α_j , such as the spectral density of q_t , also fail to recognize the influence of ε_t in determining cycles, so much of the analysis in Wen (2005) is not concerned with cycles in the NBER sense but rather with periodic cycles.

Harding and Pagan (2002) set out a framework in which the dating of cycles through turning points can be formally analyzed. Denoting the level of economic activity as Q_t , the turning points in Q_t and $q_t = \ln Q_t$ are identical and Harding and Pagan showed that it was the DGP of either ΔQ_t or Δq_t that contains all the information needed to describe the cycle in q_t . In particular, if one thinks of a linear model for Δq_t as in (1.1) then it is natural to summarize the DGP of Δq_t (and q_t) with the following parameters:

1. Long-run growth in output (μ)
2. The volatility of output growth, $\sigma_{\Delta q}$ (defined as proportional to the standard deviation of Δq_t)
3. Parameters α_j describing any serial correlation in output growth

This model is quite a good description of GDP for many countries - see Pagan (1999). Consequently, it is not surprising that the data generating process (DGP) for Δq_t , when quantified using estimates of μ, α_1, α_2 , is capable of producing a good description of many of the features of the average business cycle for a number of countries, even though the coefficients are such that the roots of the difference equation are not complex i.e. there is no periodic cycle.

It is useful to think about questions regarding the business cycle in terms of the three sets of parameters given above. Such an analysis can be qualitative or quantitative. Thus, on a qualitative level, it might be expected that a rise in μ , a fall in σ , and a reduction in positive serial correlation would lead to longer cycles. Quantitatively, once one has set out a DGP for Δq_t , it is possible to simulate data from the chosen model and to ask if the simulated characteristics are a good match with those seen in the data. There are now computer programs written in the GAUSS and MATLAB languages that can be used to automatically generate the statistics that can be used to perform such an analysis.²

As defined above a business cycle relates to the level of economic activity, q_t and turning points in it. An alternative perspective which is also common is to examine fluctuations in activity rather than a cycle, and this is often interpreted as examining the $var(\Delta q_t)$. There is a connection between the two views in that $var(\Delta q_t)$ depends upon σ and α_j , but the business cycle also emphasizes the long-run growth component of $E(\Delta q_t)$ and the serial correlation in Δq_t . Indeed, $E(\Delta q_t)$

²These are available at www.ncer.edu.au/ and were used to generate cycle information in this paper.

is very important of the nature of the business cycle. Nevertheless, because both approaches focus upon the DGP of Δq_t , what we learn about one of them can often be transferred to the other.

A number of specific questions will be addressed in this paper. First, on a general level, we want to examine the question of whether the presence of inventories is a major contributor to the business cycle. Second, there are some specific questions regarding the U.S. cycle (and fluctuations) that have arisen in recent literature which will be explored and analyzed both generally and with the model above. One of these, which came out of the experience of the long expansion of the 1990's, is whether the business cycle has become longer i.e. whether the time between successive peaks (or troughs) has become longer. Qualitatively, if the GDP growth rate was described by (1.1) we would know that this would occur if the long run growth rate of GDP increases, the volatility of GDP growth decreases, or the degree of positive correlation in growth rates lessens. McConnell and Perez-Quiros (2000) found that the volatility in the growth rate in U.S. GDP seemed to shift after the mid 1980's and this observation also seems to be true for many other countries around the world (although the date of this shift varies). Thus, such lower volatility should lead to a longer cycle.

The causes for this decline have been much debated and are surveyed and critiqued in Stock and Watson (2002). When this feature was observed it was natural that one look at what changes were taking place in the economy which might lead to such an outcome. Since there had been great advances in inventory control methods, in particular the development of "just in time" philosophies relating to production, it seemed possible that this might be a source of the changes e.g. see Kahn et al (2002).³

To understand how inventories may have affected GDP growth, we need to build a model that is capable of being quantified and which can be used to investigate what type of cycles are generated when there are inventories in the system and when there are not. The model chosen is an extension of that in Humphreys et al (2001). It sees the objectives of firms as attempting to balance the costs of keeping raw material stocks in line with output, and finished goods stocks in

³Recent studies that have looked at the role of inventory management advances and the decline in the volatility of output growth include Ahmed, Levin and Wilson (2001), Blanchard and Simon (2001), Cecchetti, Flores-Lagunes and Krause (2006), Herrera and Pesavento (2005), Iacoviello, Schiantarelli and Schuh (2006), Irvine and Schuh (2003), Kahn, McConnell and Perez-Quiros (2002), Kim and Nelson (1999), McCarthy and Zakrajsek (2002), and Ramey and Vine (2003).

line with sales, with the extra costs incurred by rapid adjustment in output and purchases of raw materials. Because of the presence of raw materials it has some additional driving forces, such as the level of raw material prices, as well as the traditional one of the sales of finished goods. The model also allows for a number of other shocks such as productivity and various cost shocks associated with inventories. The model yields optimal decision rules for value added (GDP), raw material stocks and finished good stocks. We utilize quarterly data to obtain some estimates of the parameters of the model and then conduct a number of experiments designed to explore whether inventory control methods or other forces were responsible for the reduction in the volatility of GDP growth. Our purpose is to study the role of inventories in the business cycle and not to model the goods producing sector in the US, so that the quantitative analysis we engage in is simply to get some idea of the magnitude of the parameters for our model.

Section 2 of the paper examines various features of the DGP of GDP growth and looks at the characteristics of cycles in aggregate and goods-sector GDPs. Section 3 sets out our extended version of the Humphreys et al model and the Euler equations. Section 4 estimates the parameters of this model. Section 5 conducts a number of experiments with it to gain some appreciation of what the role of inventories in the business cycle might be. Our conclusions are that on average inventories play a relatively small role in the business cycle and the volatility of GDP growth owes little to changing inventory technology.

2. Some Analysis of U.S. GDP Growth and Cycles

It seems useful to re-examine the relation of inventories and the business cycle by utilizing the approach and techniques of the view of cycles described above i.e. as one reflecting turning points. We begin by thinking of aggregate economic activity as being usefully summarized by GDP - see Burns and Mitchell (1946,p 72) for an early statement and NBER (2003) for a recent one. However, since inventories are principally used in the production of goods, determining their role in the business cycle would seem to begin by splitting GDP into its goods and non-goods (services and structures) components, and then looking at the cycle in the goods component. Designating Q^g and $Q^s = Q - Q^g$ as the goods and non-goods components of GDP respectively, a log linearization produces the approximation

$$\Delta q_t = \omega^g \Delta q_t^g + (1 - \omega^g) \Delta q_t^s$$

where ω^g is the average of the ratio $\frac{Q_t^g}{Q_t}$. Over 1947/1-2005/4 this average was .31 with a standard deviation of .02. The first and last observations on it were .32 and .36 respectively. So it has been a fairly stable ratio, and this points to the fact that the characteristics of the DGP of Δq_t are usefully analyzed by looking at the characteristics of the DGPs of Δq_t^g and Δq_t^s , as summarized by (1.1). Consequently, Table 1 shows AR(2) processes fitted to those two series, as well as aggregate GDP growth, over the period 1947/1-2005/4.

	Goods	Non-goods	Agg
μ	.0087	.0082	.0084
α_1	.105	.374	.303
α_2	.104	.084	.083
σ	.018	.007	.009

It is clear that the long-run growth in all quantities (μ) is effectively the same. However the fact that the serial correlation patterns and volatilities are very different will mean that the cycles in the goods and non-goods contributions to GDP are potentially quite different. It is interesting that the t ratios for α_j for Δq_t^g are less than 1.61, so that there is virtually no serial correlation in it. The familiar first order serial correlation seen in aggregate US GDP therefore stems from the non-goods sector.

The different characteristics noted above work in different directions when it comes to determining the impact upon cycles. The much higher volatility in goods GDP growth will mean a shorter cycle in it than for services. Offsetting this however is the lower positive serial correlation in goods output, as simulations in Harding and Pagan (2002) point to this producing a longer cycle. Consequently, with the two factors operating in opposite directions, the relative length of the cycles is indeterminate, although the very large differences in volatility suggest a much shorter cycle in goods GDP. Table 2 shows that this is indeed the case. In this table the evidence presented on the cycle is the average duration and amplitude of phases (expansions and contractions), the cumulative loss or gain in output during the phase, the shapes of the phases (through the excess statistic described in Harding and Pagan (2002), although here the divisor is not the duration of the phase but rather the cumulative gain or loss in output), and the variability

of phases as summarized by their coefficients of variation - see Engel et al (2005) for a description of the latter measures. It is clear that the cycle in goods GDP is much shorter than that in the non-goods sector. Since most attention has been paid to cycles in the level of economic activity as measured by variables such as GDP it is interesting to examine the cycle in the sub-set of GDP that relates to goods. One of the striking features of the business cycle measured with GDP is that movements in this do not signal a recession in 2001, as there was a sequence of alternating positive and negative quarterly growth rates, with the positive ones always offsetting the negative ones, meaning that there was no decline in the level of GDP for two quarters. In contrast there was a clear recession in the goods sector, starting in 2000/3 and finishing in 2001/3. Indeed it is always the case that recessions in the goods sector have been stronger and longer than those in aggregate GDP. It might be thought that this comes from a declining contribution to aggregate GDP of goods, but, as the ratios $\frac{Q_t^g}{Q_t}$ presented earlier show, the opposite has happened in the chain-weighted data. Of course in nominal terms the ratio may well have declined since the relative price of goods to non-goods has almost certainly declined.

Table 2: US Business Cycle Characteristics,				
Goods, Services and Aggregate GDP : 1947/1-2005/4				
	Goods	Non-goods	Agg	
Dur Con	3.3	3.2	2.8	
Dur Expan	12.3	36.5	20.7	
Amp Con	-4.2	-1.2	-2.0	
Amp Expan	18.0	32.9	21.7	
Cum Con	-9.4	-2.7	-3.7	
Cum Expan	180.4	736.0	366	
Excess Con	-.066	-.033	.003	
Excess Expan	.108	.099	.14	
CV Dur Con	.370	.342	.439	
CV Dur Expan	.814	.537	.833	
CV Amp Con	-.586	-.443	-.567	
CV Amp Expan	.733	.521	.680	

To explore the impact of inventories upon the cycle we focus upon Δq_t^g in this paper, since it is the behavior of this series which will be affected by the presence of inventories. On a broad level it is worth exploring the question of how inventories might affect the goods cycle by examining how the DGP of Δq_t^g is built up, i.e., what determines the parameters and the shocks in the AR(2) process. It would not be expected that the presence of inventories would affect long-run growth μ but we might expect that there could be some impact upon the dynamic response of q_t^g to shocks. Since ε_t will be built up from all the shocks of the macro-economy, and one of these could be inventory cost shocks, this is another way in which inventories could affect the cycle.

Continuing with this theme, we begin with the identity

$$X_t = Y_t - \Delta N_t \tag{2.1}$$

where X_t is the level of gross sales, Y_t is the level of gross output, and N_t is the level of finished goods inventories. Then, if the holding of finished goods inventories is important to the cycle, we would expect that the DGP of $x_t = \ln X_t$ would be different to that of $y_t = \ln Y_t$. We constructed series on X_t and N_t and then found Y_t from the identity (2.1) -see Appendix A. It is worth mentioning that the X_t we construct is not that referred to as " final sales" in the NIPA. The latter is

$$Z_t = Q_t + \Delta N_t + \Delta M_t,$$

where M_t is the level of raw material inventories. Many investigations of inventories use Z_t e.g. Wen (2005), but it is clear that, when raw materials are present, X_t and Z_t may be very different, and we cannot use the latter as a proxy for the former when attempting to quantify a model. It would seem that the series we use for X_t is only available for the goods sector of GDP, at least on a quarterly basis.

Table 3 looks at the characteristics of the DGP of each of the series $\Delta x_t, \Delta y_t$ over the period 1959/3-2005/4, as summarized by an AR(2), and these may be compared to the values for the parameters of the AR(2) process for Δq_t^g in Table 1 (although it should be noted that different sample sizes are used, reflecting data availability)

	Δx_t	Δy_t	Δu_t	Δv_t
μ	.0081	.0081	.0082	-.0007
α_1	.383	.332	.275	.276
α_2	-.013	.002	.081	.138
σ	.014	.015	.017	.006

It is clear that the presence of inventories in the US has meant very little change to the characteristics of the AR(2) DGP of sales and output growth, and, hence one would expect only small differences in their cycles. In fact recessions in both measures of activity are about the same length of four quarters while expansions differ on average by two quarters. This therefore suggests a relatively minor role, on average, for finished goods inventories in the US cycle. What is particularly striking about Table 3 however is that the DGPs of y_t and x_t are

much closer to that for aggregate and service GDP than they are for goods GDP and this suggests that any effect of inventories must come via the transformation leading from y_t to q_t^g , and we therefore turn to this relationship.

Value added Q_t^g is the goods contribution to GDP, and this relates to Y_t through the identities

$$\begin{aligned} Q_t^g &= Y_t - V_t U_t \\ \Delta M_t &= D_t - U_t \end{aligned}$$

where U_t is usage of raw materials, D_t is deliveries of raw materials, V_t is the relative price of raw materials to the price of output and M_t is the stock of raw materials.⁴ Thus inventories of raw materials may modify the cycle in y_t , producing a different one to that in q_t^g , and we have already seen that these DGPs are quite different.⁵

Further insight into the nature of the DGP for Δq_t^g is then available from the log linear approximation

$$\Delta q_t^g = \omega_y \Delta y_t - \omega_{uv} \Delta v_t - \omega_{uv} \Delta u_t,$$

where ω_y and ω_{uv} are the averages of $\frac{Y_t}{Q_t^g}$ and $\frac{V_t U_t}{Q_t^g}$ respectively. Taking the weights as the sample averages over 1959/1-1983/4. Fig 1 plots Δq_t^g and the approximation above, and it is clear that the match is very good. Thus it is apparent that the DGP of Δq_t^g might differ from that of Δy_t either because of the impact of Δv_t or of Δu_t . To assess this in more detail first look at Table 3. There is less dynamic structure to both the Δu_t and Δv_t processes and thus we would expect a reduction in the serial correlation of the Δq_t^g from that of Δx_t , i.e., the presence of raw material usage acts to reduce the dependence that would come from final sales. Moreover, the volatility of Δq_t^g will derive from the joint behavior of both Δu_t and Δv_t and, once again, this can induce higher volatility in Δq_t^g that was not present in final sales or output. Indeed, if it was the case that $\omega_y + \omega_{uv} = 1$, and usage of raw materials was a constant fraction of output, then we would have

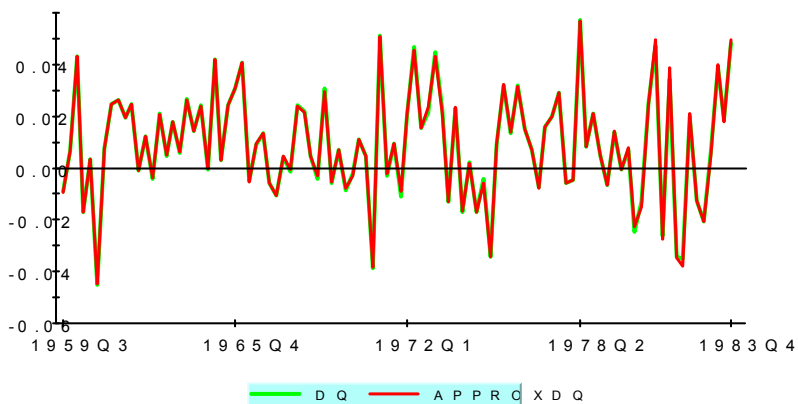
$$\Delta q_t^g = \Delta y_t - \omega_{uv} \Delta v_t,$$

so that $var(\Delta q_t^g) > var(\Delta y_t)$ due to the volatility in Δv_t . This point emphasizes that the volatility in raw material prices will be a determinant of that in GDP.

⁴There are missing elements in this definition such as energy usage and imports. Also we have assumed that the price of materials used is the same as that of the stock of raw materials.

⁵We can measure the quantities in the identities above in the following way. First, V_t is taken to be the implicit price deflator for raw materials used in the private business sector divided by the implicit price deflator for goods GDP. Second, U_t may then be recovered from $\frac{Y_t - Q_t^g}{V_t}$.

Fig 1 Goods GDP Growth and Approximation from Linearize Output, Goods GDP and Materials Identity



Of course the analysis above has not specifically isolated the impact of inventories. As mentioned earlier cost shocks in inventories may be important in affecting the volatility of Δu_t . Moreover, holding raw material inventories may enable one to smooth usage and to take advantage of discounts so that costs are lower for any level of V_t . Ultimately, we need to develop a model describing how Δq_t^g is influenced by the presence of inventories, in particular one that allows for both finished goods and raw materials inventories. We turn to this task in the next section

As well as being concerned with how the cycle in activity might reflect the presence of inventories we are also interested in debates in recent years that have pointed to changing cycle characteristics and which have suggested a role for inventories in these. To focus on that debate it is useful to briefly consider the behavior of some of these variables over time. Table 4 shows the DGP for Δq_t^g over two periods of time; the particular partition being selected as representing the view that the decline in $\sigma_{\Delta q}$ occurred around the mid 1980s. It is clear from this table that there have also been some changes in the serial correlation structures and a large drop in volatility. The latter has been remarked upon a good deal for aggregate GDP but it also shows up in goods GDP and, within the individual components discussed above, there has also been a halving of volatility for Δx_t (from .018 to .008) but not in Δv_t (which rises from .006 to .007) .

	1947/1-1983/4	1984/1-2005/4
μ	.0083	.0096
α_1	.102	.106
α_2	.074	.260
$\sigma_{\Delta q}$.021	.011

It is also worth looking at the characteristics of Δy_t . Unfortunately, due to data limitations the first period is much shorter than in Table 4. But the same features are apparent, so that it would seem as if the longer business cycles in recent years probably do not stem from inventories.

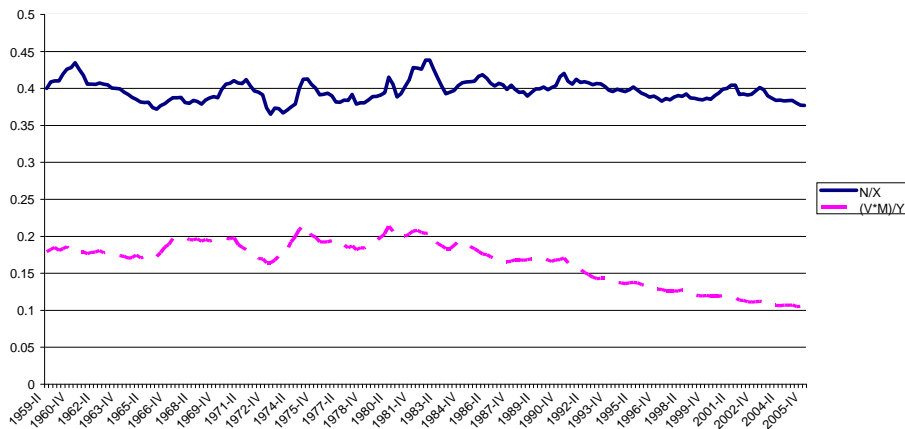
	1959/1-1983/4	1984/1-2005/4
μ	.0080	.0082
α_1	.306	.471
α_2	.022	-.114
$\sigma_{\Delta q^g}$.019	.009

Since one of the possible reasons for the changes in volatility has been given as changing inventory behavior, notably the ability to economize on inventories with new technology, it is worth looking at the ratios $\frac{N_t}{X_t}$ and $\frac{V_t M_t}{Y_t}$ over time.⁶ Fig 2 gives a plot of these. It is clear that there has been almost no change in the first ratio, but the second has declined by about 50% after 1984, which is of course a substantial decline. This again points to the potential importance of raw materials in looking at changes in cycles and, for the US, these are likely to have been more significant than finished goods inventories. Whether changes in the levels of inventories that are held can in fact explain changes in the cycle is

⁶We look at $\frac{V_t M_t}{Y_t}$ since a change in V_t would be expected to change $\frac{M_t}{Y_t}$ and so a change in the latter may simply reflect a response to relative price changes rather than a technological change. Of course even this ratio may not fully control for such an effect.

Note that essentially the same pattern occurs for $\frac{M_t}{Y_t}$. It too declines after 1984 by about 45%, so it declines slightly less precipitously.

Fig 2 Ratios of N to X and V*M to Y



a different matter, and once again points to the need to develop a model that explains Δq_t^g and which formally incorporates raw materials.

3. The Model and its Euler Equations

The model of the representative firm that we use is an extension of the one developed by Humphreys, et al.(2001). The model in Humphreys et al. has the advantage that it is a model of inventories broken down by stage of fabrication and thus distinguishes between finished goods or "output" inventories and materials and supplies or "input" inventories. The latter includes work in progress inventories as well; hereafter, we use the term materials inventories to refer to the sum of materials and supplies and work in progress inventories. The model thus permits an analysis of the role that each type of inventory stock plays in the production and sales process. This is an important advantage of the model as finished goods and materials inventories may have played very different roles in the reduction of the volatility of GDP growth.⁷ Figure 2.2 reported that the materials-output ratio had declined about 50% since the early eighties, but the

⁷Iacoviello et al (2006) also develop a model where both input and output inventories are held in the goods sector. Their model, however, differs from the one developed here in the motivation for holding input and output inventories, the use of different definitions of input and output inventories, and the lack of a distinction between gross output and value added.

finished goods-sales ratio had remained about constant. This suggests that, to the extent that improved inventory management techniques have had a role to play in reducing the volatility of GDP growth, materials inventories may have been more important than finished goods inventories. Further, "just-in-time" techniques which have become more widely used in recent years are more applicable to materials inventory management than to that of finished goods.

3.1. The Production Function

We begin with a specification of the short-run production function, which is assumed to be Cobb-Douglas

$$\begin{aligned} Y_t &= F(L_t, U_t, \epsilon_{yt}) \\ &= L_t^{\gamma_1} U_t^{\gamma_2} e^{\epsilon_{yt}} \end{aligned} \tag{3.1}$$

where $\gamma_1 + \gamma_2 < 1$, which implies strict concavity of the production function in materials usage and labor. Here, Y_t is output, L_t is labor input, U_t is materials usage, and ϵ_{yt} is a technology shock. Note that U_t is the *flow* of materials used in the production process. When production and inventory decisions are made, the capital stock is assumed to be taken as given by the firm and to be growing at a constant rate, which will be captured by a deterministic trend in the empirical work. Finally, the firm is assumed to purchase intermediate goods (work-in-process) from outside suppliers rather than producing them internally.⁸ Thus, intermediate goods are analogous to raw materials so work-in-process inventories can be lumped together with materials inventories. Because Y_t is gross output, we refer to equation (3.1) as the gross production function.

3.2. The Cost Structure

The firm's total cost structure consists of three major components: labor costs, inventory holding costs, and materials costs. This section describes each component.

⁸To allow for production of intermediate goods within the firm requires extending the production function to incorporate joint production of final and intermediate goods. This extension is a substantial modification of the standard production process that we leave for future work.

3.2.1. Labor Costs

Labor costs are

$$\begin{aligned} LC_t &= W_t L_t + W_t \tilde{A}(L_t, L_{t-1}) \\ &= W_t L_t + W_t A(\Delta l_t - \Delta \bar{l}) L_{t-1} \end{aligned} \quad (3.2)$$

where $\Delta l_t = \Delta \log L_t \approx \frac{\Delta L_t}{L_{t-1}}$ is the growth rate of labor and $\Delta \bar{l}$ is the steady state growth rate of labor. The first component, $W_t L_t$, is the standard wage bill where W_t is the real wage rate. The second component, $\tilde{A}(L_t, L_{t-1})$, is an adjustment cost function intended to capture the hiring and firing costs associated with changes in labor inputs. The adjustment cost function has the usual properties: Specifically,

$$\begin{aligned} A' &\gtrless 0 \quad \text{as } \Delta l_t \gtrless \Delta \bar{l} \\ A(0) &= A'(0) = 0 \quad A'' > 0 \end{aligned}$$

Adjustment costs on labor accrue whenever the growth rate of the firm's labor force is different from the steady state growth rate. Further, adjustment costs exhibit rising marginal costs.

3.2.2. Inventory Holding Costs

Inventory holding costs for finished goods inventories are:

$$HC_t^N = \Phi^N(N_{t-1}, X_t, \epsilon_{nt-1}) = \delta_1 \left(\frac{N_{t-1}}{X_t} \right)^{\delta_2} X_t + \delta_3 N_{t-1} + \epsilon_{nt-1} N_{t-1} \quad (3.3)$$

$$\delta_1 > 0 \quad \delta_2 < 0 \quad \delta_3 > 0$$

where ϵ_{nt} is the white noise innovation to finished goods inventory holding costs. Inventory holding costs consist of two basic components. One, $\delta_1 \left(\frac{N_{t-1}}{X_t} \right)^{\delta_2} X_t$, captures the idea that, given sales, higher inventories reduce costs in the form of lost sales because they reduce stockouts. The other, $\delta_3 N_{t-1}$, captures the idea that higher inventories raise costs because they raise holding costs

in the form of storage costs, insurance costs, etc.⁹ The effects of technological advances that improve inventory management methods can be captured, for example, through a change in δ_2 and perhaps changes in other parameters as well.¹⁰

Inventory holding costs for materials and supplies inventories are:

$$\begin{aligned}
 HC_t^M &= \Phi^M(V_{t-1}M_{t-1}, Y_t, \epsilon_{mt-1}) \\
 &= \tau_1 \left(\frac{V_{t-1}M_{t-1}}{Y_t} \right)^{\tau_2} Y_t + \tau_3 V_{t-1}M_{t-1} + \epsilon_{mt-1} V_{t-1}M_{t-1}
 \end{aligned} \tag{3.4}$$

$$\tau_1 > 0 \quad \tau_2 < 0 \quad \tau_3 > 0$$

where ϵ_{mt} is the white noise innovation to materials inventory holding costs. As above, there are two basic components: One, $\tau_1 \left(\frac{V_{t-1}M_{t-1}}{Y_t} \right)^{\tau_2} Y_t$, captures the idea that, given output, higher inventories reduces costs in the form of lost output because they reduce stockouts and disruptions to the production process. The other, $\tau_3 V_{t-1}M_{t-1}$, captures the idea that higher inventories raises costs because they raise holding costs in the form of storage costs, insurance costs, etc.⁴ Note that materials inventory holding costs depend on production, rather than sales. This is because stocking out of materials inventories entails costs associated with production disruptions – lost production, so to speak – that are distinct from the costs associated with lost sales. Lost production may be manifested by reduced

⁹These two components underlie the rationale for the quadratic inventory holding costs in the standard linear-quadratic model. The formulation above separates the components and assumes constant elasticity functional forms which facilitates log-linearization around constant steady states.

We assume that the firm minimizes discounted expected costs and thereby abstract from market structure issues. See Bils and Kahn (2000) for a model that deals with market structure issues and also utilizes a constant elasticity specification of the benefits of holding finished goods inventories, though the benefits are embedded on the revenue side of the firm.

¹⁰Observe that (3.3) implies a "target stock" of finished goods inventories that minimizes finished goods inventory holding costs. Assuming $\epsilon_{nt-1} = 0$, the target stock, N_t^* , is

$$N_t^* = - \left(\frac{\delta_3}{\delta_1 \delta_2} \right)^{\frac{1}{\delta_2 - 1}} X_t$$

so that the implied target stock is proportional to sales, which is analogous to the target stock assumed in the standard linear-quadratic model.

productivity or failure to realize production plans.¹¹

The finished goods and material inventory holding costs differ because the firm holds the two inventory stocks for different reasons. The firm stocks finished goods inventories to guard against random demand fluctuations, but it stocks materials inventories to guard against random fluctuations in productivity, materials prices and deliveries, and other aspects of production. Although sales and production are highly positively correlated, they differ enough at high frequencies to justify different specifications.

3.2.3. Materials Costs

Finally, we turn to the cost of purchasing materials and supplies. We assume that the real cost of purchasing materials and supplies is given by

$$\begin{aligned}
 MC_t &= V_t D_t + V_t \tilde{\Gamma}(V_t D_t, Y_t) = V_t D_t + V_t \Gamma\left(\frac{V_t D_t}{Y_t}\right) D_t & (3.5) \\
 &= V_t D_t \left[1 + \Gamma\left(\frac{V_t D_t}{Y_t}\right)\right] \\
 \Gamma' &\geq 0 & \Gamma'' &> 0 \\
 \Gamma(0) &= 0 & \Gamma\left(\frac{\overline{VD}}{\overline{Y}}\right) &= 0 & \Gamma'\left(\frac{\overline{VD}}{\overline{Y}}\right) &= 0
 \end{aligned}$$

where V_t is a real "base price" for raw materials. The term, $V_t D_t$, is the value of purchases and deliveries valued at the base price. The term, $V_t \tilde{\Gamma}(V_t D_t, Y_t)$, represents a premium that may need to be paid over and above the base price to undertake the level of purchases and deliveries, D_t . It is assumed to rise at an increasing rate with the amount purchased and delivered.

Two cases may be distinguished:

1. Increasing Marginal Cost: $\Gamma' > 0$. In this case, the firm faces a rising supply price for materials purchases. When purchases are high relative to current stocks, the firm thus experiences increasing marginal costs due to higher

¹¹Similarly, observe that (3.4) implies a "target stock" of materials and supplies inventories that minimizes materials and supplies inventory holding costs. Assuming $\epsilon_{mt-1} = 0$, the target stock, $V_t M_t^*$, is

$$V_t M_t^* = - \left(\frac{\tau_3}{\tau_1 \tau_2}\right)^{\frac{1}{\tau_2 - 1}} Y_t$$

so that the implied target stock is proportional to output.

premia that must be paid to acquire materials more quickly. A rationale for such a rising supply price is that the firm is a monopsonist in the market for materials. This is most likely to occur when materials are highly firm or industry specific and the firm or industry is a relatively large fraction of market demand.¹² The rising marginal cost of course gives rise to the “smoothing” of purchases.

2. Constant Marginal Cost: $\Gamma' = 0$. In this case, the firm is in effect a price taker in competitive input markets and is able to purchase all the raw materials it needs at the prevailing market price.

3.3. Cost Minimization

Assume that the representative firm takes sales (X_t) and factor prices (V_t and W_t) as exogenous. The firm’s optimization problem is to minimize the discounted present value of total costs (TC),

$$E_0 \sum_{t=0}^{\infty} \beta^t TC_t = E_0 \sum_{t=0}^{\infty} \beta^t (LC_t + HC_t + MC_t), \quad (3.6)$$

where $\beta = (1 + r)^{-1}$ is the discount factor. The constraints are the production function, (3.1), and the two laws of motion governing inventory stocks. The law for finished goods inventories is

$$\Delta N_t = Y_t - X_t \quad (3.7)$$

and the law for materials and supplies inventories is

$$\Delta M_t = D_t - U_t \quad (3.8)$$

The firm chooses $\{L_t, U_t, Y_t, M_t, N_t, D_t\}_{t=0}^{\infty}$ to minimize equation (3.6) subject to the constraints (3.1), (3.7), and (3.8).

¹²This is analogous to the literature on adjustment cost models for investment in plant and equipment where external adjustment costs are imposed in the form of a rising supply price for capital goods. See, e.g. Gould (1968) or Abel(1979).

3.4. Optimality Conditions

Assume that, when the representative firm makes decisions, current values of exogenous variables are in its information set. Then, define the following shares and ratios

$$\begin{aligned} S_{L,t} &= \frac{W_t L_t}{Y_t} & S_{U,t} &= \frac{V_t U_t}{Y_t} & R_{N,t} &= \frac{N_t}{X_t} & R_{M,t} &= \frac{V_t M_t}{Y_t} \\ R_{D,t} &= \frac{V_t D_t}{Y_t} & R_{Y,t} &= \frac{Y_t}{X_t} & R_{\Lambda_3,t} &= \frac{\Lambda_{3t}}{V_t} \end{aligned} \quad (3.9)$$

Further, recall that lower case letters are the logarithms of an upper case letter, so, for example, $l_t = \log L_t$, and thus the growth rate of a variable is $\Delta l_t = \Delta \log L_t \approx \frac{\Delta L_t}{L_{t-1}}$. Then, in Appendix B, we show that the optimality conditions may be written as

$$\begin{aligned} 1 + \Lambda_{1t} \gamma_1 S_{L,t}^{-1} + A'(\Delta l_t - \Delta \bar{l}) + \beta E_t (1 + \Delta w_{t+1}) A(\Delta l_{t+1} - \Delta \bar{l}) \\ - \beta E_t (1 + \Delta w_{t+1}) (1 + \Delta l_{t+1}) A'(\Delta l_{t+1} - \Delta \bar{l}) = 0 \end{aligned} \quad (3.10)$$

$$\Lambda_{1t} \gamma_2 S_{U,t}^{-1} + R_{\Lambda_3,t} = 0 \quad (3.11)$$

$$1 + \Gamma(R_{D,t}) + R_{D,t} \Gamma'(R_{D,t}) - R_{\Lambda_3,t} = 0 \quad (3.12)$$

$$(1 - \tau_2) \tau_1 (R_{M,t-1} (1 - \Delta y_t))^{\tau_2} - R_{D,t}^2 \Gamma'(R_{D,t}) - \Lambda_{1t} - \Lambda_{2t} = 0 \quad (3.13)$$

$$\Lambda_{2t} - \beta E_t \Lambda_{2t+1} + \beta E_t \delta_2 \delta_1 (R_{N,t} (1 - \Delta x_{t+1}))^{\delta_2 - 1} + \beta \delta_3 + \beta \epsilon_{nt} = 0 \quad (3.14)$$

$$\begin{aligned} R_{\Lambda_3,t} - \beta E_t (1 + \Delta v_{t+1}) R_{\Lambda_3,t+1} + \beta E_t \tau_2 \tau_1 (R_{M,t} (1 - \Delta y_{t+1}))^{\tau_2 - 1} + \beta \tau_3 \\ + \beta \epsilon_{mt} = 0 \end{aligned} \quad (3.15)$$

$$\Delta y_t = \gamma_1 \Delta l_t + \gamma_2 \Delta u_t + \epsilon_{yt} - \epsilon_{yt-1} \quad (3.16)$$

$$R_{N,t} - R_{N,t-1} (1 - \Delta x_t) - R_{Y,t} + 1 = 0 \quad (3.17)$$

$$R_{M,t} - (1 + \Delta v_t - \Delta y_t) R_{M,t-1} - R_{D,t} + S_{U,t} = 0 \quad (3.18)$$

where Λ_{1t} , Λ_{2t} , and Λ_{3t} are Lagrangian multipliers associated with (3.1), (3.7) and (3.8) respectively. We assume that in steady state the factor input shares, $S_{U,t}$ and $S_{L,t}$, the ratios $R_{N,t}$, $R_{M,t}$, $R_{D,t}$, $R_{Y,t}$, and $R_{\Lambda_3,t}$, and the growth rates of variables are constants. The non-stochastic steady state conditions are:

$$1 + \bar{\Lambda}_1 \gamma_1 \bar{S}_L^{-1} = 0 \quad (3.19)$$

$$1 + \bar{\Lambda}_1 \gamma_2 \bar{S}_U^{-1} = 0 \quad (3.20)$$

$$\bar{R}_{\Lambda_3} = 1 \quad (3.21)$$

$$(1 - \tau_2) \tau_1 [\bar{R}_M (1 - \Delta \bar{y})]^{T_2} - \bar{\Lambda}_1 - \bar{\Lambda}_2 = 0 \quad (3.22)$$

$$(1 - \beta) \bar{\Lambda}_2 + \beta \delta_2 \delta_1 [\bar{R}_N (1 - \Delta \bar{x})]^{\delta_2 - 1} + \beta \delta_3 = 0 \quad (3.23)$$

$$[1 - \beta (1 + \Delta \bar{v})] + \beta \tau_2 \tau_1 [\bar{R}_M (1 - \Delta \bar{y})]^{T_2 - 1} + \beta \tau_3 = 0 \quad (3.24)$$

$$\gamma_1 \Delta \bar{l} + \gamma_2 \Delta \bar{u} = \Delta \bar{y} \quad (3.25)$$

$$1 + \Delta \bar{x} \bar{R}_N = \bar{R}_Y \quad (3.26)$$

$$\bar{R}_D + (\Delta \bar{v} - \Delta \bar{y}) \bar{R}_M = \bar{S}_U \quad (3.27)$$

We then log-linearize the optimality conditions around the constant steady state values. On notation, note that a "hat" above an upper case letter denotes a log-deviation, and a "hat" above a lower case letter denotes a simple (i.e., non-logarithmic) deviation. So, for example, the log-deviation of the level of sales is $\hat{X}_t = \log X_t - \log \bar{X}_t$, and the simple deviation of the growth rate of sales is $\Delta \hat{x}_t = \Delta x_t - \Delta \bar{x}$. Similar notation applies to other variables. The log-linearized optimality conditions are then

$$\hat{S}_{L,t} - \hat{\Lambda}_{1t} + \varphi \Delta \hat{l}_t - \beta \varphi E_t \Delta \hat{l}_{t+1} = 0 \quad (3.28)$$

$$\hat{S}_{U,t} - \hat{\Lambda}_{1t} + \hat{R}_{\Lambda_3,t} = 0 \quad (3.29)$$

$$\eta \hat{R}_{D,t} - \hat{R}_{\Lambda_3,t} = 0 \quad (3.30)$$

$$\mu_2 \bar{R}_M (1 - \Delta \bar{y}) \left[\hat{R}_{M,t-1} - \Delta \hat{y}_t \right] + \eta \bar{R}_D \hat{R}_{D,t} + \bar{\Lambda}_1 \hat{\Lambda}_{1t} + \bar{\Lambda}_2 \hat{\Lambda}_{2t} = 0 \quad (3.31)$$

$$\bar{\Lambda}_2 \hat{\Lambda}_{2t} - \beta \bar{\Lambda}_2 E_t \hat{\Lambda}_{2t+1} + \beta \mu_1 E_t \left(\hat{R}_{N,t} - \Delta \hat{x}_{t+1} \right) + \beta \epsilon_{nt} = 0 \quad (3.32)$$

$$\hat{R}_{\Lambda_3,t} - \beta E_t \left[\Delta \hat{v}_{t+1} + \hat{R}_{\Lambda_3,t+1} \right] + \beta \mu_2 E_t \left[\hat{R}_{M,t} - \Delta \hat{y}_{t+1} \right] + \beta \epsilon_{mt} = 0 \quad (3.33)$$

$$\Delta \hat{y}_t = \gamma_1 \Delta \hat{l}_t + \gamma_2 \Delta \hat{u}_t + \epsilon_{yt} - \epsilon_{yt-1} \quad (3.34)$$

$$\hat{R}_{N,t} - (1 - \Delta \bar{x}) \hat{R}_{N,t-1} + (1 - \Delta \bar{x}) \Delta \hat{x}_t - \frac{\bar{R}_Y}{\bar{R}_N} \hat{R}_{Y,t} = 0 \quad (3.35)$$

$$\hat{R}_{M,t} - (1 + \Delta \bar{v} - \Delta \bar{y}) \left[\hat{R}_{M,t-1} + \Delta \hat{v}_t - \Delta \hat{y}_t \right] - \frac{\bar{R}_D}{\bar{R}_M} \hat{R}_{D,t} + \frac{\bar{S}_U}{\bar{R}_M} \hat{S}_{U,t} = 0 \quad (3.36)$$

where

$$\begin{aligned} \mu_1 &= (\delta_2 - 1) \delta_2 \delta_1 \left[\bar{R}_N (1 - \Delta \bar{x}) \right]^{\delta_2 - 1}, \\ \mu_2 &= (\tau_2 - 1) \tau_2 \tau_1 \left[\bar{R}_M (1 - \Delta \bar{y}) \right]^{\tau_2 - 1}, \\ \varphi &= A''(0) > 0 \quad \text{and} \quad \eta = \Gamma''(\bar{R}_D) \bar{R}_D^2 > 0. \end{aligned}$$

Now, recall that the definition of GDP is

$$Q_t^g = Y_t - V_t U_t \quad (3.37)$$

Then

$$\frac{Q_t^g}{Y_t} = R_{Q,t} = 1 - \frac{V_t U_t}{Y_t} \quad (3.38)$$

and this is related to the utilization share, $S_{U,t} = \frac{V_t U_t}{Y_t}$, through

$$R_{Q,t} = 1 - S_{U,t}. \quad (3.39)$$

The steady state solution for $R_{Q,t}$ will be

$$\bar{R}_Q = 1 - \bar{S}_U. \quad (3.40)$$

A log-linear approximation then yields

$$\bar{R}_Q \hat{R}_{Q,t} = -\bar{S}_U \hat{S}_{U,t}. \quad (3.41)$$

Finally, recalling again that a lower case letter is the logarithm of an upper case letter, the log-deviations of the shares and ratios and the simple deviations of the growth rates will be

$$\begin{aligned} \hat{S}_{L,t} &= \log S_{L,t} - \log \bar{S}_L = \log W_t + \log L_t - \log Y_t - \log \bar{S}_L \\ &= w_t + l_t - y_t - \bar{s}_L, \\ \hat{R}_{N,t} &= \log N_t - \log X_t - \log \bar{R}_N = n_t - x_t - \bar{r}_N, \\ \Delta \hat{x}_t &= \Delta x_t - \Delta \bar{x}, \text{ etc.} \end{aligned}$$

Substituting these expressions into the optimality conditions and using straightforward substitutions to eliminate l_t , y_t , u_t and d_t , as well as the multipliers, the optimality conditions reduce to three Euler equations in q_t^g , n_t , and m_t . These are are:

$$\begin{aligned} &\tilde{a}_0 - \beta \varphi \frac{\bar{R}_Q}{\bar{S}_L} E q_{t+1}^g + \left[\frac{\gamma_1}{\gamma_2} + \kappa_1 + \eta \frac{\bar{S}_L}{\bar{R}_D} \right] \frac{\bar{R}_Q}{\bar{S}_L} q_t^g - \varphi \frac{\bar{R}_Q}{\bar{S}_L} q_{t-1}^g \quad (3.42) \\ &- \beta \varphi \kappa_2 \frac{\bar{R}_N}{\bar{R}_Y} E_t n_{t+1} + \{ \kappa_2 [\kappa_1 + \beta \varphi (1 - \Delta \bar{x})] + \eta \kappa_3 - \kappa_4 \} \frac{\bar{R}_N}{\bar{R}_Y} n_t \\ &- \{ \varphi \kappa_2 + [\kappa_1 \kappa_2 + \eta \kappa_3] (1 - \Delta \bar{x}) - \kappa_4 \} \frac{\bar{R}_N}{\bar{R}_Y} n_{t-1} + \varphi (1 - \Delta \bar{x}) \kappa_2 \frac{\bar{R}_N}{\bar{R}_Y} n_{t-2} - \eta \frac{\bar{R}_M}{\bar{R}_D} m_t \\ &+ \eta \frac{\bar{R}_M}{\bar{R}_D} [1 + \Delta \bar{v} - \Delta \bar{y}] m_{t-1} - \beta \varphi \kappa_2 \frac{1}{\bar{R}_Y} E_t x_{t+1} + [\kappa_2 \kappa_1 + \eta \kappa_3 - \kappa_4] \frac{1}{\bar{R}_Y} x_t - \varphi \kappa_2 \frac{1}{\bar{R}_Y} x_{t-1} \\ &- \beta \varphi \frac{\gamma_2}{\gamma_1} v_{t+1} + \left[\frac{\gamma_2}{\gamma_1} \kappa_1 - \eta \kappa_6 \right] v_t - \varphi \frac{\gamma_2}{\gamma_1} v_{t-1} + w_t + \beta \frac{\varphi}{\gamma_1} E_t \epsilon_{yt+1} - \frac{1}{\gamma_1} \kappa_1 \epsilon_{yt} + \frac{\varphi}{\gamma_1} \epsilon_{yt-1} = 0 \end{aligned}$$

$$\begin{aligned}
& \tilde{b}_0 + \beta \left[\frac{1}{\gamma_2} - \eta\kappa_5 \right] \bar{R}_Q E_t q_{t+1}^g - \left[\frac{1}{\gamma_2} - \eta\kappa_5 \right] \bar{R}_Q q_t^g \quad (3.43) \\
& - \beta \kappa_8 \frac{\bar{R}_N}{\bar{R}_Y} E_t n_{t+1} + \left\{ (1 + \beta(1 - \Delta\bar{x})) \kappa_8 \frac{\bar{R}_N}{\bar{R}_Y} + \beta\mu_1 \right\} n_t \\
& - \kappa_8 (1 - \Delta\bar{x}) \frac{\bar{R}_N}{\bar{R}_Y} n_{t-1} + \beta\eta\bar{R}_M\kappa_5 E_t m_{t+1} + \left\{ \beta\kappa_7 - \eta\bar{R}_M\kappa_5 [1 + \beta(1 + \Delta\bar{v} - \Delta\bar{y})] \right\} m_t \\
& - [\kappa_7 - \eta\bar{R}_M\kappa_5 (1 + \Delta\bar{v} - \Delta\bar{y})] m_{t-1} - \beta \left\{ \kappa_8 \frac{1}{\bar{R}_Y} + \mu_1 \right\} E_t x_{t+1} \\
& + \kappa_8 \frac{1}{\bar{R}_Y} x_t + \beta\eta\kappa_5\kappa_6\bar{R}_D E_t v_{t+1} + \left\{ \beta\kappa_7 - \eta\kappa_5\kappa_6\bar{R}_D \right\} v_t - \kappa_7 v_{t-1} + \beta\epsilon_{nt} = 0
\end{aligned}$$

$$\begin{aligned}
& \tilde{c}_0 + \beta\eta\frac{\bar{R}_Q}{\bar{R}_D} E_t q_{t+1}^g - \eta\frac{\bar{R}_Q}{\bar{R}_D} q_t^g - \beta \left\{ \mu_2 - \eta\kappa_3 \right\} \frac{\bar{R}_N}{\bar{R}_Y} E_t n_{t+1} \quad (3.44) \\
& + \left\{ \beta\mu_2 - \eta(1 + \beta(1 - \Delta\bar{x})) \kappa_3 \right\} \frac{\bar{R}_N}{\bar{R}_Y} n_t + \eta(1 - \Delta\bar{x}) \kappa_3 \frac{\bar{R}_N}{\bar{R}_Y} n_{t-1} \\
& - \beta\eta\frac{\bar{R}_M}{\bar{R}_D} E_t m_{t+1} + \left\{ \beta\mu_2 + \eta[1 + \beta(1 + \Delta\bar{v} - \Delta\bar{y})] \frac{\bar{R}_M}{\bar{R}_D} \right\} m_t \\
& - \eta(1 + \Delta\bar{v} - \Delta\bar{y}) \frac{\bar{R}_M}{\bar{R}_D} m_{t-1} - \beta \left[\mu_2 - \eta\kappa_3 \right] \frac{1}{\bar{R}_Y} E_t x_{t+1} - \eta\kappa_3 \frac{1}{\bar{R}_Y} x_t \\
& - \beta [1 + \eta\kappa_6] E_t v_{t+1} + \left\{ \beta(1 + \mu_2) + \eta\kappa_6 \right\} v_t + \beta\epsilon_{mt} = 0
\end{aligned}$$

where

$$\begin{aligned}
\kappa_1 &= 1 + (1 + \beta)\varphi & \kappa_2 &= \frac{1}{\gamma_1} - \frac{1}{S_L} \\
\kappa_3 &= 1 - \frac{1}{\bar{R}_D} & \kappa_4 &= \frac{1}{1 - \bar{R}_Q} \\
\kappa_5 &= 1 - \frac{1}{\gamma_2} \frac{1 - \bar{R}_Q}{\bar{R}_D} & \kappa_6 &= 1 - \frac{1 - \bar{R}_Q}{\bar{R}_D} \\
\kappa_7 &= \bar{R}_M (1 - \Delta\bar{y}) \mu_2 & \kappa_8 &= \kappa_7 + \frac{1}{\gamma_2} \bar{R}_Q - \eta(1 - \bar{R}_D) \kappa_5
\end{aligned}$$

In the expressions above the constant terms, \tilde{a}_0 , \tilde{b}_0 , and \tilde{c}_0 depend on the constant steady state shares, ratios and growth rates. These are the Euler equations which will be used in the empirical work of the next section.

4. Estimation of the Model

The model above is that of a representative firm. To apply the model to the goods sector as a whole, we assume that the representative firm behaves as if it is vertically integrated, so that it is representative of the whole goods sector of the economy. The representative firm holds materials inventory stocks which it uses in conjunction with labor (and capital) to produce output of finished goods, which it adds to finished goods inventories. The finished goods inventories may be held by manufacturers, wholesalers or retailers. In effect, we treat the representative firm as managing the inventory stocks of finished goods whether they are held on the shelves of the manufacturer, the wholesaler, or the retailer.

In order to answer some of the questions raised about inventories in our initial section we need to be able to calibrate the model of the preceding section. Our objective here is not to provide a detailed fit to observed data but to gain some appreciation of the magnitude of the parameters that would be needed to reproduce outcomes in the U.S. economy over the period 1959/1-1983/4. Ideally one would like to have begun with 1947/1 but quarterly data was not available on M_t and N_t over that earlier period. Our decision to fit the model over a short period was also driven by the fact that over this period ratios such as ω^g , $\frac{N_t}{X_t}$ and $\frac{V_t M_t}{Y_t}$ were reasonably stable. Since we were interested in what might account for changes occurring after 1983/4 it seemed best not to estimate the model with data after that point. Once calibrated the model can then be used to explore some of the questions raised in the introduction.

Let

$$z_t = \begin{bmatrix} n_t \\ q_t^g \\ m_t \end{bmatrix}, \xi_t = \begin{bmatrix} x_t \\ w_t \\ v_t \end{bmatrix}, \varepsilon_t = \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{nt} \\ \varepsilon_{mt} \end{bmatrix},$$

so that ξ_t are the observable shocks and ε_t are unobservable ones. All data were detrended by fitting linear polynomials. The data suggests that x_t and w_t are $I(1)$, with ADF(4) test values of -2.2 and -1.6 respectively. First order serial correlation was also present in Δx_t and Δw_t , with first order serial correlation coefficients of .33 and .38 respectively. v_t seemed closer to stationarity, having an AR(2) of the

form

$$v_t = 1.21v_{t-1} - .29v_{t-2} + .0053\varepsilon_t^v.$$

The v_t process is a very persistent one but the sharp rise in oil prices (and associated raw material prices) in 1974 had a major effect upon the unit root tests. Because of this we decided to treat it as $I(0)$ and following the DGP above. The ε_t were taken to be AR(1) processes.

The parameters in the Euler equations can be divided into three groups:

$$(I) \theta_1 = [\beta, \gamma_1, \gamma_2, \bar{R}_Y, \bar{R}_D, \bar{R}_M, \bar{R}_N] \quad (4.1)$$

$$(II) \theta_2 = [\tau_3, \delta_3, \bar{\Lambda}_1, \bar{\Lambda}_2], \quad (4.2)$$

$$(III) \theta_3 = [\tau_1, \tau_2, \delta_1, \delta_2, \varphi, \eta, \sigma_y, \sigma_n, \sigma_m, \rho_y, \rho_n, \rho_n], \quad (4.3)$$

where σ_j and ρ_j are the standard deviations and AR(1) parameters of the unobservable shocks $\varepsilon_y, \varepsilon_n$ and ε_n . θ_1 is either pre-set - $\beta = .99, \gamma_1 = .22, \gamma_2 = .66$ - or estimated using sample means of the ratios. γ_j were set because the absence of capital in the model makes it difficult to estimate the parameters of a production process.¹³

The four parameters in the second set θ_2 are found from the four equations corresponding to the steady state conditions (3.20) and (3.21)-(3.24), once values for θ_3 are available. This means it is necessary to estimate θ_3 . From the optimality conditions in (3.28) and (3.33) it is evident that τ_1, τ_2, δ_1 and δ_2 enter only through μ_1 and μ_2 , so only two of these parameters can be identified. Consequently we set $\delta_1 = \tau_1 = 1$.

In our model derivations we have effectively assumed that there are two cointegrating relations among the observable $I(1)$ variables given by the ratios $R_{N,t} = \frac{N_t}{X_t}$ and the raw material usage share $S_{U,t} = \frac{V_t U_t}{Y_t}$. Labour usage was not in our data set but we assume that the labour share $S_{L,t}$ is $I(0)$. If these three variables are $I(0)$ then (3.10) shows Λ_{1t} is $I(0)$, (3.11) makes $R_{\Lambda_{3t}}$ $I(0)$, (3.15) has $R_{M,t}$ as $I(0)$, (3.12) makes $R_{D,t}$ $I(0)$, (3.13) makes Λ_{2t} $I(0)$, (3.14) makes $R_{N,t}$ $I(0)$ and (3.17) makes $R_{Y,t}$ an $I(0)$ variable. This means that we have five $I(1)$ variables in z_t and ξ_t and two common permanent components in the form of x_t and w_t . Thus another common permanent component is needed and we identify this

¹³If we had set $\gamma_2 = .71$ to reflect the share of raw materials as measured in our data then it seems virtually impossible to allow for a role for capital, as a realistic share for labour would mean that the sum of the raw material and labour shares would exceed unity. We found that setting γ_j below the values above resulted in a substantial decline in the likelihood so these values seemed a reasonable compromise.

as the technology shock, so that $\rho_y = 1$. These arguments leave the parameters to be estimated as $[\tau_2, \delta_2, \varphi, \eta, \sigma_y, \sigma_n, \sigma_m, \rho_n, \rho_m]$.

The parameters were estimated by MLE using Dynare Version 3.04.6 written by S. Adjemian, M. Juillard and O. Kamenik. Estimates and t ratios are in Table 6. The implied estimates of τ_3 and δ_3 are .21 and .13 respectively.¹⁴

	τ_2	δ_2	φ	η	σ_y
Est	-.039	-.055	1.22	7.4	.008
t	3.5	3.5	4.1	7.9	11.8
	σ_n	σ_m	ρ_n	ρ_m	
Est	.005	.010	.94	.82	
t	4.3	4.7	9.4	12.2	

Although we were not trying to produce a model of the goods sector it is interesting to look at the adequacy of the model in re-producing the characteristics of the empirical DGP of Δq_t^g . For the parameter values in Table 6 the implied standard deviation for Δq_t^g , $\sigma_{\Delta q^g}$, is .0238, versus the value of .021 in the data. The test statistic that these are different is 2.1 so that the model seems to produce a reasonable fit to the variance of Δq_t^g , although a little too high. The parameter estimates imply a first serial correlation coefficient in Δq_t^g of .07, which is less than that of the data, although not significantly different (t ratio of -.9).

5. Experiments with the Model

5.1. Analysis of Fluctuations

We adopt the calibration above as the "base model" and then ask how $\sigma_{\Delta q^g}$ varies with changes in selected parameters of the model. In particular we are interested in what happens as δ_2 and τ_2 change so as to mean that less finished goods inventories and raw materials are held as a ratio of sales and output respectively.

¹⁴If we treated v_t as $I(1)$ the parameter estimates are similar but the log likelihood would be 1776 versus the 1780 obtained when it is assumed to be $I(0)$. The standard deviation of Δq_t would also be slightly higher at .0241. These results led us to prefer the assumption that v_t was $I(0)$.

Consider, for example, a decline in the absolute value of τ_2 , which shifts the materials inventory holding cost function. Intuitively, such a decline captures the idea that computerization, just-in time procedures, or other technological advances in inventory management techniques imply that the firm can experience the same level of lost production with a smaller level of materials inventories, given the level of output. Or, alternatively, a given materials inventory/output ratio will generate a smaller amount of lost production. A similar interpretation applies to a decline in the absolute value of δ_2 . To compute values of Δq_t^g , when a parameter changes, we reverse the estimation strategy, and now solve for ratios such as R_N and R_M as functions of the estimated model parameters. Also of interest is the magnitude of the impact of changes in the volatility of the observed and unobserved shocks.

$|\delta_2|$ was arbitrarily reduced by 10% while $|\tau_2|$ was reduced by 20%. The latter produced a decline in the $\frac{VM}{Y}$ ratio that roughly matches what was seen over the period 1984/1-2005/4. Similarly the reductions in standard deviations of all shocks was set to 50%, as that was roughly what happened to the observable shocks over that period. So these experiments are about how we might have expected fluctuations to have changed over the second period given that the volatility reduction in observed shocks was matched by that in the unobserved ones. The experiments involving reductions in the standard deviations of shocks also give some insight into what the main sources of fluctuations would be. Finally, we present an experiment in which δ_2 and τ_2 are just one-hundredth of the values in Table 6. Such a reduction makes inventories extremely expensive to hold and it emulates a situation where inventories are not present in the system. Table 7 gives the results of these experiments.

	$\sigma_{\Delta q^g}$ (cuts in parameters)
Base	.0238
.9 δ_2	.0238
.8 τ_2	.0233
.5 σ_x	.0206
.5 σ_w	.0237
.5 σ_v	.0228
.5 σ_n	.0237
.5 σ_m	.0233
.5 σ_y	.0190
.01(δ_2, τ_2)	.0195

It is clear that, if the change in inventory technology can be thought of as involving a change in the magnitude of τ_2 , so that less raw materials are an optimal choice, then this produces only slightly smaller fluctuations in goods GDP. Hence this cannot be the source of the reduced volatility in Δq_t^g . Decomposing the variance of Δq_t^g into the shocks 48% is due to the technology shock, 6% from the raw materials inventory shock, 11% from the raw material price shock and 33% from the sales shock. Neither wages nor the final inventory shocks are important.

If the volatility of all shocks was reduced by 50%, $\sigma_{\Delta q^g}$ would become .012 which is very close to the actual reduction over the 1984/1-2005/4 period - see Table 4. If only the observed ones were reduced, $\sigma_{\Delta q^g}$ would only have dropped to .019 so that the unobservable shocks are critical to explain this phenomenon. Specifically we will need a large decline in the volatility of technology shocks.

An alternative viewpoint, expressed in Kahn et al (2002), is that the information set of firms may have changed as a consequence of computerization. In particular it may be that sales are now known with greater accuracy. To assess this we considered an experiment in which it was assumed that only $\{x_{t-j-1}\}_{j=0}^{\infty}$ rather than $\{x_t\}_{j=0}^{\infty}$ was known in the period 1959/1-1983/4. In the post-1983/4 period however x_t was taken to be part of the information set. To conduct this experiment the model was re-estimated using the new information set and the implied $\sigma_{\Delta q^g}$ was found to be .0252. Hence the reduction in the volatility of Δq_t^g as a result of improved information about sales is very small, since the base case in Table 7 represents what it would be with the expanded information.

In light of these results it is useful to consider the debate over whether mone-

tary policy had an impact on $\sigma_{\Delta q^g}$. One might expect that this effect would work through sales and, although the decline in the volatility of the latter has made a contribution, it would not have led to the observed decline in volatility if technology shocks had not changed as well. Thus it is hard to see monetary policy as being the major driving force in the reduction in the volatility in the goods sector.

5.2. Analysis of Cycles

Whilst the nature of the business cycle depends upon the volatility of Δq_t^g it also depends upon the mean of this process and the nature of any serial correlation in it. Consequently, the experiments above were repeated to determine their effects upon the cycle in q_t^g . Table 8 shows how the durations and amplitudes of expansions and contractions in q_t^g would change. It should be noted that over the period of estimation, 1959/1-1983/4, the duration of contractions and expansions were 3.43 and 9.45 quarters, and so the length of expansions using the estimated model parameters (the "base" simulation) is quite close to that actually observed.

	Dur(Con)	Dur (Exp)	Amp(Con)	Amp(Ex)
Base	3.93	9.74	-5.30	16.14
.9 δ_2	3.92	9.76	-5.28	16.13
.8 τ_2	3.91	9.88	-5.16	16.09
.5 σ_x	3.33	11.34	-3.64	15.27
.5 σ_w	3.92	9.69	-5.26	16.04
.5 σ_v	3.91	10.13	-5.03	16.17
.5 σ_n	3.93	9.74	-5.30	16.14
.5 σ_m	3.91	9.82	-5.18	16.07
.5 σ_y	3.85	10.36	-4.38	15.63
.01(δ_2, τ_2)	3.93	11.82	-4.20	16.72

Given what we know about the connection between cycle length and the volatility of Δq_t the results in Table 8 are largely predictable by the outcomes in Table 7. An exception occurs for the relative effects of the experiments involving a reduction in sales and technology shock volatilities. Here the cycle becomes longer with the first experiment, even though the volatility decrease was less than in the

second experiment. This shows that the degree of serial correlation in Δq_t^g is also important for cycle outcomes. The final experiment shows that the presence of inventories in the system does create more cycles although only a few quarters more in length. However, examination of the coefficient of variation of the amplitudes of expansions shows this is about 11% higher with the changed parameter values so that the variability of expansions does depend upon the presence of inventories in the system. Overall, the importance of inventories to the average cycle is limited, even though it may be that for particular cycles their presence has a greater effect. It should be noted that in no case are there complex roots in the Δq_t^g process and so no periodic cycles.

6. Conclusion

We have developed a model of the optimal holding of finished goods and raw material input inventories by a goods producing firm and have used it to analyze a number of questions that have come up about the role of inventories. It was shown that changes in inventory technology have little effect upon the volatility of GDP goods sector growth and that inventories have only a small effect upon the average duration of expansions and contractions. To show the latter we looked at business cycles in terms of the turning points in the level of goods GDP, which is a very different perspective to the traditional work on inventory cycles that looked at periodic cycles in activity. The model we develop allows for a role for raw material prices in producing cycles and we found that the latter did have some importance, although the main driver of the business cycle remained technology variations.

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8. Appendix A: Data Description

In the model above we assumed that the representative firm behaves as if it is vertically integrated, so that it is representative of the goods sector of the whole economy. We assumed that the representative firm holds materials inventory stocks which it uses in conjunction with labor (and capital) to produce output of finished goods, which it adds to finished goods inventories. The finished goods inventories may then be held by manufacturers, wholesalers or retailers. In effect, we treat the representative firm as managing the inventory stocks of finished goods

whether they are held on the shelves of the manufacturer, the wholesaler, or the retailer.

Accordingly, we construct an aggregate stock of finished goods inventories by summing the real value of finished goods inventories in manufacturing, wholesale trade and retail trade. The aggregate stock of materials inventories is constructed by adding up the materials and supplies and work in progress inventories held by manufacturers.¹⁵ Value added or GDP is the real value of GDP for the goods sector of the economy. We constructed an approximate measure of gross output for the goods sector by summing gross sales for the manufacturing and trade sectors of the economy and finished goods inventory investment for that sector. The data are quarterly, seasonally-adjusted, (2000) chain-weighted series in billions of dollars, and cover the period 1959/1 through 2005/4. GDP is of course the flow of value added over the quarter, and inventories are measured as end-of-quarter stocks.

The series on W is the ratio of the average hourly earnings for goods producing industries divided by the implicit price deflator for sales of the business sector. V is found by dividing the implicit price deflator for input inventories by the implicit price deflator for the sales of the business sector.

8.1. Appendix B: Derivation of Optimality Conditions

Assume that current values are in the information set. Recall again that lower case letters are the logarithms of an upper case letter, so, for example, $l_t = \log L_t$, and thus the growth rate of a variable is $\Delta l_t = \Delta \log L_t \approx \frac{\Delta L_t}{L_{t-1}}$. Then, using the functional form assumptions, the optimality conditions are:

$$W_t + \Lambda_{1t}\gamma_1 \frac{Y_t}{L_t} + W_t A' (\Delta l_t - \Delta \bar{l}) + \beta E_t W_{t+1} A (\Delta l_{t+1} - \Delta \bar{l}) \quad (8.1)$$

$$-\beta E_t W_{t+1} \frac{L_{t+1}}{L_t} A' (\Delta l_{t+1} - \Delta \bar{l}) = 0$$

$$\Lambda_{1t}\gamma_2 \frac{Y_t}{U_t} + \Lambda_{3t} = 0 \quad (8.2)$$

$$V_t \left[1 + \Gamma\left(\frac{V_t D_t}{Y_t}\right) + \frac{V_t D_t}{Y_t} \Gamma'\left(\frac{V_t D_t}{Y_t}\right) \right] - \Lambda_{3t} = 0 \quad (8.3)$$

¹⁵Note that there are no materials and supplies and work in progress inventories in wholesale trade or retail trade.

$$(1 - \tau_2) \tau_1 \left(\frac{V_{t-1} M_{t-1}}{Y_t} \right)^{\tau_2} - \left(\frac{V_t D_t}{Y_t} \right)^2 \Gamma' \left(\frac{V_t D_t}{Y_t} \right) - \Lambda_{1t} - \Lambda_{2t} = 0 \quad (8.4)$$

$$\Lambda_{2t} - \beta E_t \Lambda_{2t+1} + \beta E_t \delta_2 \delta_1 \left(\frac{N_t}{X_{t+1}} \right)^{\delta_2 - 1} + \beta \delta_3 + \beta \epsilon_{nt} = 0 \quad (8.5)$$

$$\Lambda_{3t} - \beta E_t \Lambda_{3t+1} + \beta E_t \tau_2 \tau_1 V_t \left(\frac{V_t M_t}{Y_{t+1}} \right)^{\tau_2 - 1} + \beta V_t \tau_3 + \beta V_t \epsilon_{mt} = 0$$

$$Y_t = (L_t)^{\gamma_1} (U_t)^{\gamma_2} e^{\epsilon_{yt}} \quad (8.6)$$

$$N_t - N_{t-1} - Y_t + X_t = 0 \quad (8.7)$$

$$M_t - M_{t-1} - D_t + U_t = 0 \quad (8.8)$$

Re-arranging terms, we have that

$$1 + \Lambda_{1t} \gamma_1 \frac{Y_t}{W_t L_t} + A' (\Delta l_t - \Delta \bar{l}) + \beta E_t \frac{W_{t+1}}{W_t} [A (\Delta l_{t+1} - \Delta \bar{l})] \quad (8.9)$$

$$-\beta E_t \frac{W_{t+1}}{W_t} \frac{L_{t+1}}{L_t} A' (\Delta l_{t+1} - \Delta \bar{l}) = 0$$

$$\Lambda_{1t} \gamma_2 \frac{Y_t}{V_t U_t} + \frac{\Lambda_{3t}}{V_t} = 0 \quad (8.10)$$

$$1 + \Gamma \left(\frac{V_t D_t}{Y_t} \right) + \frac{V_t D_t}{Y_t} \Gamma' \left(\frac{V_t D_t}{Y_t} \right) - \frac{\Lambda_{3t}}{V_t} = 0 \quad (8.11)$$

$$(1 - \tau_2) \tau_1 \left(\frac{V_{t-1} M_{t-1}}{Y_{t-1}} \frac{Y_{t-1}}{Y_t} \right)^{\tau_2} - \left(\frac{V_t D_t}{Y_t} \right)^2 \Gamma' \left(\frac{V_t D_t}{Y_t} \right) - \Lambda_{1t} - \Lambda_{2t} = 0 \quad (8.12)$$

$$\Lambda_{2t} - \beta E_t \Lambda_{2t+1} + \beta E_t \delta_2 \delta_1 \left(\frac{N_t}{X_t} \frac{X_t}{X_{t+1}} \right)^{\delta_2 - 1} + \beta \delta_3 + \beta \epsilon_{nt} = 0 \quad (8.13)$$

$$\frac{\Lambda_{3t}}{V_t} - \beta E_t \frac{V_{t+1}}{V_t} \frac{\Lambda_{3t+1}}{V_{t+1}} + \beta E_t \tau_2 \tau_1 \left(\frac{V_t M_t}{Y_t} \frac{Y_t}{Y_{t+1}} \right)^{\tau_2 - 1} + \beta \tau_3 + \beta \epsilon_{mt} = 0 \quad (8.14)$$

$$\left(\frac{Y_t}{Y_{t-1}} \right) = \left(\frac{L_t}{L_{t-1}} \right)^{\gamma_1} \left(\frac{U_t}{U_{t-1}} \right)^{\gamma_2} e^{\epsilon_{yt} - \epsilon_{yt-1}} \quad (8.15)$$

$$\frac{N_t}{X_t} - \frac{N_{t-1}}{X_{t-1}} \frac{X_{t-1}}{X_t} - \frac{Y_t}{X_t} + 1 = 0 \quad (8.16)$$

$$\frac{V_t M_t}{Y_t} - \frac{V_t}{V_{t-1}} \frac{V_{t-1} M_{t-1}}{Y_{t-1}} \frac{Y_{t-1}}{Y_t} - \frac{V_t D_t}{Y_t} + \frac{V_t U_t}{Y_t} = 0 \quad (8.17)$$

Now, define the growth rate of, for example, X_t , by $X_t = (1 + \Delta x_t) X_{t-1}$, and similarly for other variables. Further, use the approximation, $\frac{1}{1 + \Delta x_t} \approx 1 - \Delta x_t$ to re-write the optimality conditions as

$$1 + \Lambda_{1t} \gamma_1 \frac{Y_t}{W_t L_t} + A'(\Delta l_t - \Delta \bar{l}) + \beta E_t (1 + w_{t+1}) A(\Delta l_{t+1} - \Delta \bar{l}) - \beta E_t (1 + w_{t+1}) (1 + l_{t+1}) A'(\Delta l_{t+1} - \Delta \bar{l}) = 0 \quad (8.18)$$

$$\Lambda_{1t} \gamma_2 \frac{Y_t}{V_t U_t} + \frac{\Lambda_{3t}}{V_t} = 0 \quad (8.19)$$

$$1 + \Gamma\left(\frac{V_t D_t}{Y_t}\right) + \frac{V_t D_t}{Y_t} \Gamma'\left(\frac{V_t D_t}{Y_t}\right) - \frac{\Lambda_{3t}}{V_t} = 0 \quad (8.20)$$

$$(1 - \tau_2) \tau_1 \left(\frac{V_{t-1} M_{t-1}}{Y_{t-1}} (1 - \Delta y_t) \right)^{\tau_2} - \left(\frac{V_t D_t}{Y_t} \right)^2 \Gamma'\left(\frac{V_t D_t}{Y_t}\right) - \Lambda_{1t} - \Lambda_{2t} = 0 \quad (8.21)$$

$$\Lambda_{2t} - \beta E_t \Lambda_{2t+1} + \beta E_t \delta_2 \delta_1 \left(\frac{N_t}{X_t} (1 - \Delta x_{t+1}) \right)^{\delta_2 - 1} + \beta \delta_3 + \beta \epsilon_{nt} = 0 \quad (8.22)$$

$$\frac{\Lambda_{3t}}{V_t} - \beta E_t (1 + \Delta v_{t+1}) \frac{\Lambda_{3t+1}}{V_{t+1}} + \beta E_t \tau_2 \tau_1 \left(\frac{V_t M_t}{Y_t} (1 - \Delta y_{t+1}) \right)^{\tau_2 - 1} + \beta \tau_3 \quad (8.23)$$

$$+ \beta \epsilon_{mt} = 0$$

$$\Delta y_t = \gamma_1 \Delta l_t + \gamma_2 \Delta u_t + \epsilon_{yt} - \epsilon_{yt-1} \quad (8.24)$$

$$\frac{N_t}{X_t} - \frac{N_{t-1}}{X_{t-1}} (1 - \Delta x_t) - \frac{Y_t}{X_t} + 1 = 0 \quad (8.25)$$

$$\frac{V_t M_t}{Y_t} - (1 + \Delta v_t) (1 - \Delta y_t) \frac{V_{t-1} M_{t-1}}{Y_{t-1}} - \frac{V_t D_t}{Y_t} + \frac{V_t U_t}{Y_t} = 0 \quad (8.26)$$

Then, using the definitions of the shares and ratios stated in (3.9), the optimality conditions are (3.10)-(3.18). Log-linearizing these conditions yields the log-linearized optimality conditions stated in (3.28)-(3.36) together with the steady state conditions (3.19)-(3.27).