On the Efficacy of Fourier Series Approximations for Pricing European and Digital Options

A S Hurn
K A Lindsay
A J McClelland

Working Paper #90
January 2013
ON THE EFFICACY OF FOURIER SERIES APPROXIMATIONS FOR PRICING EUROPEAN AND DIGITAL OPTIONS

A. S. HURN*, K. A. LINDSAY†, AND A. J. MCCLELLAND‡

Abstract. This paper investigates several competing procedures for computing the price of European and digital options in which the underlying model has a characteristic function that is known in at least semi-closed form. The algorithms for pricing the options investigated here are the half-range Fourier cosine series, the half-range Fourier sine series and the full-range Fourier series. The performance of the algorithms is assessed in simulation experiments which price options in a Black-Scholes world where an analytical solution is available and for a simple affine model of stochastic volatility in which there is no closed-form solution. The results suggest that the half-range sine series approximation is the least effective of the three proposed algorithms. It is rather more difficult to distinguish between the performance of the half-range cosine series and the full-range Fourier series. There are however two clear differences. First, when the interval over which the density is approximated is relatively large, the full-range Fourier series is at least as good as the half-range Fourier cosine series, and outperforms the latter in pricing out-of-the-money call options, in particular with maturities of three months or less. Second, the computational time required by the half-range Fourier cosine series is uniformly longer than that required by the full-range Fourier series for an interval of fixed length. Taken together, these two conclusions make a strong case for the merit of pricing options using a full-range range Fourier series as opposed to a half-range Fourier cosine series.

Key words. Fourier transform, Fourier series, characteristic function, option price

AMS subject classifications. 62P05, 91G20, 91G60

*School of Economics and Finance, Queensland University of Technology
†School of Mathematics and Statistics, University of Glasgow and School of Economics and Finance, Queensland University of Technology
‡Sydney Numerix
1. Introduction. This paper investigates several competing procedures for computing the price of European and digital options. This problem has received new impetus in recent years with the development of particle filtering approaches to the estimation of stochastic volatility models. This procedure allows time-series returns data to be combined with cross-sectional options data at each point in time [11], [3], [5], [10]. The computational complexity of this approach is driven by the requirement that each function evaluation of a search algorithm involves the calculation of around 1 billion model option prices when estimating simple models with reasonable amounts of data. On a desktop PC a realistic outcome is one function evaluation per day, which implies one full model estimation every 6 months or so thereby rendering any estimation exercise infeasible. The innovations that allow this kind of computation to be reduced to a matter of hours as opposed to months stems from the fact that a particle filtering algorithm lends itself to parallel computation. Indeed, in the parlance of computer theorists the problem is “embarrassingly parallel” and computation in this kind of environment is becoming increasingly of interest [6], [7].

While major advances are being made in computational speed due to parallelisation on graphical processing units (GPU), the fact remains that the major numerical effort in estimating the parameters of stochastic volatility models resides in the computational load arising from the need to price large numbers of options with various strike prices and maturities, not to mention the management of daily changes in the value of the risk-free rate of interest. The ability to price options accurately and efficiently cannot be overestimated and re-visiting this issue is timely.

Various strategies have been proposed for calculating the price of option contracts from knowledge of the conditional characteristic function of the underlying model. It is an important fact that a surprisingly large number of models have a semi-closed expression for their conditional characteristic function. For example, the identification of the conditional characteristic function for multivariate affine models with/without jump processes leads to the solution of a family of ordinary differential equations, albeit in the complex plane. In view of the Levy-Khintchine theorem, the identification of the conditional characteristic function for Levy processes is expressed in terms of various integrals with respect to the Levy measure.

The most commonly used techniques for taking advantage of a known conditional characteristic function have at their core the application of the Fast Fourier Transform (FFT). The most well documented of these approaches is due to Carr and Madan [4] who construct an expression for the price of a European call option in terms of an integral over the characteristic function. This integral, which has an oscillatory kernel, is computed by an application of the FFT. Borak, Detlefsen and Hardle [2] apply the FFT strategy and demonstrate its efficacy by comparison with Monte Carlo simulation for a variety of models. Lord, Fang, Bervoets and Oosterlee [13] and Kwok, Leung and Wong [12] demonstrate how Fourier’s convolution theorem in combination with the FFT can be used to price certain exotic options from knowledge of the conditional characteristic function of the price of the underlying asset. A different approach pioneered by Fang and Oosterlee [8] uses the characteristic function to directly approximate the marginal transitional probability density of returns by a Fourier cosine series. More recently Zhang, Grzelak and Oosterlee [14] demonstrate how this methodology can be extended to the pricing of early-exercise commodity options under the Ornstein-Uhlenbeck process.

Rather than describe in detail the nuances of these various strategies, it is useful to
point out what overarching assumptions connect them. Recall that the FFT is simply a clever piece of linear algebra that reduces the arithmetical load in implementing the Discrete Fourier Transform (DFT), namely the pair of equations connecting the coefficients of a finite Fourier series with values of the underlying function and vice versa. Therefore the decision to use the FFT implicitly makes the assumption that the underlying function is periodic over an interval of finite length, in practice determined by the frequencies submitted to the characteristic function, and that the function has been approximated over the interval by a finite Fourier series. The values of Fourier coefficients calculated from the characteristic function are in error by the extent to which the finite Fourier transform\(^1\) differs from the Fourier transform.

Thus techniques using the FFT and those based on the construction of Fourier series share the same common assumptions and deficiencies. However, an important difference between an implementation using the FFT approach and one using the Fourier series approach is that the latter is parsimonious in its use of arithmetic whereas the former typically performs more arithmetic than is necessary, albeit in an efficient way. For example, if the FFT is used to determine the value of a probability density function what is recovered is the value of the function at each node of the interval, whereas all that what might be needed is the value of the probability density function over a sub-interval.

The focus of this work is on the algorithm proposed by Fang and Oosterlee \([8]\) who give a convincing demonstration of the efficacy of the Fourier cosine series. This series is more accurately called the half-range\(^2\) cosine series because the actual function to be expanded is defined only on half the interval of periodicity (or range), the function being extended to the full range as an even-valued function. Half-range cosine series usually fail to represent derivatives whereas half-range Fourier series usually fail to represent function values. While the use of the half-range Fourier cosine series is a solid idea, Fang and Oosterlee Fang and Oosterlee \([8]\) provide no motivation or explanation as to why this choice of approximating transitional density should be preferred over the half-range Fourier sine series or the full-range Fourier series for that matter. For example, intuition would suggest that the latter might perform better simply because it uses higher frequencies which in turn translate to a more rapidly converging Fourier expansion. Indeed this intuition is borne out in calculation, but of course speed is not the only criterion of relevance in assessing the efficacy of a numerical procedure.

An important but subtle difference between the half-range cosine series and full-range Fourier series approximations of density and that based on the half-range sine series is that the former assign unit probability to the interval of support when in reality probability lies outside this interval, whereas the latter imposes zero probability density at the endpoints of the interval of support in contravention of reality, but on the other hand does not assign unit probability to the interval of support. Is one approach always superior to the other or is it a case of horses for courses? Intuition might suggest the latter. For example, when pricing a call option the most important component of the pricing error comes from the exclusion of contributions from asset price exterior to the finite interval of support. Because the half-range cosine and full-range Fourier series necessarily capture unit density, intuition might suggest that these approximations provide potential compensation for this component of pricing

\(^1\)The finite Fourier transform is the integral expression defining the coefficients of a Fourier series.

\(^2\)Historically, half-range Fourier series largely arise as analytical tools for handling different types of boundary conditions when solving partial differential equations using integral transforms.
error. On the other hand intuition would suggest that the same approximations, when used to price digital options, might have a tendency to exaggerate the probability of exercise and therefore overprice this option in contrast to the half-range sine series approximation of probability density.

2. Fourier series and transform. Suppose that \( f(y) \) satisfies the Dirichlet conditions on \([a, b]\), then there are three common ways in which \( f(y) \) may be represented by a Fourier series. These are the half-range Fourier cosine series, the half-range Fourier sine series and the full-range Fourier series with their respective representations

\[
\begin{align*}
(a) \quad & \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \left( \frac{k\pi(y-a)}{b-a} \right), \\
(b) \quad & \sum_{k=1}^{\infty} b_k \sin \left( \frac{k\pi(y-a)}{b-a} \right), \\
(c) \quad & \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \left( \frac{2k\pi(y-a)}{b-a} \right) + b_k \sin \left( \frac{2k\pi(y-a)}{b-a} \right). 
\end{align*}
\]

(2.1)

The use of the term “half-range” in describing expressions (a) and (b) simply refers to the fact that the function \( f(y) \), although defined in \([a, b]\), has for the construction of the Fourier series been extended into the interval \([2a-b, a]\) as an even-valued function in the case of the half-range cosine series (so that sine contributions vanish) and as an odd-valued function in the case of the half-range sine series (so that the constant and cosine contributions vanish). Thus both half-range series are conventional Fourier series taken over the interval \([2a-b, b]\) such that the function represented by the half-range Fourier cosine series is usually not differentiable at \( x = a \), whereas that represented by the half-range Fourier sine series is usually discontinuous at \( x = a \).

In the case of the half-range cosine and sine series in expressions (2.1a) and (2.1b) respectively, the coefficients \( a_k \) \((k \geq 0)\) and \( b_k \) \((k \geq 1)\) are calculated from the function \( f(y) \) via the formulae

\[
\begin{align*}
a_k &= \frac{2}{b-a} \int_{a}^{b} f(y) \cos \left( \frac{k\pi(y-a)}{b-a} \right) dy \\
b_k &= \frac{2}{b-a} \int_{a}^{b} f(y) \sin \left( \frac{k\pi(y-a)}{b-a} \right) dy.
\end{align*}
\]

(2.2)

When expressed in terms of the exponential function the coefficients \( a_k \) and \( b_k \) become

\[
\begin{align*}
a_k &= \frac{2}{b-a} \int_{a}^{b} \Re \left[ f(y) \exp \left( \frac{k\pi iy}{b-a} \right) \exp \left( \frac{-k\pi ia}{b-a} \right) dy \right], \\
b_k &= \frac{2}{b-a} \int_{a}^{b} \Im \left[ f(y) \exp \left( \frac{k\pi iy}{b-a} \right) \exp \left( \frac{-k\pi ia}{b-a} \right) dy \right].
\end{align*}
\]

(2.3)

In the case of the full-range Fourier series in expression (2.1c) the coefficients \( a_k \) \((k \geq 0)\) and \( b_k \) \((k \geq 1)\) are calculated from the function \( f(y) \) via the formulae

\[
\begin{align*}
a_k &= \frac{1}{b-a} \int_{a}^{b} f(y) \cos \left( \frac{2k\pi(y-a)}{b-a} \right) dy, \\
b_k &= \frac{1}{b-a} \int_{a}^{b} f(y) \sin \left( \frac{2k\pi(y-a)}{b-a} \right) dy.
\end{align*}
\]

(2.4)
both of which can be brought together in the single complex expression
\[ a_k + ib_k = \frac{1}{b-a} \int_a^b f(y) \exp\left( \frac{2k\pi iy}{b-a} \right) \exp\left( -\frac{2k\pi ia}{b-a} \right) dy. \] (2.5)

Suppose now that \( f(y) \) is a transitional probability density function with known characteristic function defined formally by the equation
\[ \chi(\omega) = \mathbb{E}[e^{i\omega y}] = \int_{\mathbb{R}} f(y) e^{i\omega y} dy, \] (2.6)
where \( \omega \in \mathbb{R} \) and \( \chi(0) = 1 \) irrespective of the specification of the density \( f(y) \). A necessary condition for \( f(y) \) to be a probability density function is that \( f(y) \to 0 \) as \( |y| \to \infty \), and therefore there is guaranteed to be an interval \([a, b] \) such that for all \( y \in (-\infty, a] \cup [b, \infty) \) it can be asserted that \( f(y) < \varepsilon \) for any arbitrary small positive \( \varepsilon \).

The implication of this observation is that the Fourier coefficients in equations (2.3) and (2.5) can be approximated from knowledge of the characteristic function via the respective formulae
\[ a_k \approx A_k = \frac{2}{b-a} \mathbb{R} \left[ \chi\left( \frac{k\pi}{b-a} \right) \exp\left( -\frac{k\pi ia}{b-a} \right) \right], \]
\[ b_k \approx B_k = \frac{2}{b-a} \mathbb{I} \left[ \chi\left( \frac{k\pi}{b-a} \right) \exp\left( -\frac{k\pi ia}{b-a} \right) \right], \] (2.7)
while the coefficients of the full-range Fourier series can be approximated from knowledge of the characteristic function via the formula
\[ a_k + ib_k \approx A_k + iB_k = \frac{1}{b-a} \chi\left( \frac{2k\pi}{b-a} \right) \exp\left( -\frac{2k\pi ia}{b-a} \right). \] (2.8)

The accuracy of approximations (2.7) and (2.8) is investigated in Section 6, where it is demonstrated that the error can be made arbitrarily small by choosing a suitably large interval.

3. Approximating Probability Density Functions. The quality of this practical idea is now explored for three trial probability density functions with known closed-form expressions for their characteristic functions. The first choice is the Gaussian density which may be regarded as representative of distributions with super-exponentially decaying tail density. The second and third choices are the Gamma density and the Cauchy density which are treated as representative examples of distributions with exponentially decaying and algebraically decaying tail density respectively.

3.1. Gaussian density. It is standard knowledge that the Gaussian density with mean value \( \mu \) and variance \( \sigma^2 \) has characteristic function \( \chi(\omega) = \exp\left( i\mu \omega - \sigma^2 \omega^2 / 2 \right) \) which corresponds to the equivalent specifications \( c_n = \exp(-\sigma^2 k_n^2 / 2) \cos(\mu k_n) \) and \( s_n = \exp(-\sigma^2 k_n^2 / 2) \sin(\mu k_n) \). In order to demonstrate the quality with which the true probability density \( f(x) \) can be reconstructed from a truncated Fourier series of the form of equation (2.1 c), suppose that \( \mu = \sigma = 1 \) and take the interval of support to be \([a, b] = [-3, 5] \), i.e. four standard deviations on either side of the mean value. The approximating function in this case using \( N/2 \) frequencies is
\[ \hat{f}(x) = \frac{1}{8} \left[ c_0 + 2 \sum_{n=1}^{N/2} \left( c_n \cos k_n x + s_n \sin k_n x \right) \right], \quad k_n = \frac{n\pi}{4}. \] (3.1)
Figure 3.1 illustrates the quality of this approximation using 40 frequencies \((N = 80)\) and using 4 frequencies \((N = 8)\).

![Comparison of the true Gaussian density and its approximation based on 40 frequencies (solid line, \(N = 80\)) and 4 frequencies (dashed line, \(N = 8\)).](image)

With as few as 4 frequencies it is clear that the approximating density still provides a good representation of the true density; with 40 frequencies the approximating density function is indistinguishable from the true density function. The explanation of this excellent performance stems from the property that the error bound \(F(a) + 1 - F(b)\) converges to zero super-exponentially as \(a \to -\infty\) and \(b \to \infty\).

### 3.2. Gamma density

The Gamma density with shape parameter \(\alpha\) and scale parameter \(\beta\) has probability density function and characteristic function given by the respective formulae

\[
f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad \chi(\omega) = \frac{1}{(1 - i\beta\omega)^\alpha}.
\]

(3.2)

The approximating density is identical to expression (3.1) with \(b - a = 8\) but now

\[
c_n = \frac{\cos(\alpha \tan^{-1} \beta k_n)}{(1 + \beta^2 k_n^2)^{\alpha/2}}, \quad s_n = \frac{\sin(\alpha \tan^{-1} \beta k_n)}{(1 + \beta^2 k_n^2)^{\alpha/2}}.
\]

The task is again to reconstruct the Gamma probability density function from \(\chi(\omega)\). The fact that the deviate \(Y = X/\beta\) is also Gamma distributed, in this case with shape parameter \(\alpha\) and unit scale parameter, suggest that for the Gamma distribution the appropriate comparison is between the approximating Fourier series and a Gamma distribution with unit scale parameter. Figure 3.2 illustrates the quality of this approximation for \(\Gamma(\frac{1}{2}, 1)\) for 50 frequencies (solid line, \(N = 100\)) and 20 frequencies (dashed line, \(N = 40\)). With the exception of the region very close to the origin, the true density and approximated density are not significantly different for even 20 frequencies, and with 50 frequencies the difference between the true density and the approximating density is not discernible with the exception of the origin which does not present a difficulty since the density to known to be zero there.
The quality of the approximation is again due to the fact that the error bound $1 - F(b)$ converges to zero exponentially as $b \to \infty$. Typical values for the coefficient of mean reversion (say, $\kappa = 3.0$), the mean volatility (say $\gamma = 0.02$) and the volatility of volatility (say $\sigma = 0.2$) in Heston’s model of stochastic volatility lead to a stationary distribution of volatility described by a Gamma density with shape parameter $\alpha = 2\kappa\gamma/\sigma^2 = 3.0$. A second example with $\alpha = 3$ and $\beta = 1$ is illustrated in Figure 3.3.

3.3. Cauchy density. The Cauchy density with median $\mu$ and scale parameter $\alpha$ has probability density function and characteristic function given by the respective formulae

$$f(x) = \frac{\alpha}{\pi} \frac{1}{(x-\mu)^2 + \alpha^2}, \quad \chi(\omega) = e^{i\mu\omega - \alpha |\omega|}. \quad (3.3)$$
The approximating density is again expression (3.1) with \( b-a = 20 \), \( c_n = e^{-\alpha k_n} \cos(\mu k_n) \) and \( s_n = e^{-\alpha k_n} \sin(\mu k_n) \). The task is now to reconstruct the Cauchy probability density function from \( \chi(\omega) \), and for this purpose let \( \mu = 2 \), \( \alpha = 1 \) and suppose that \( f(x) \) is treated as a function of compact support over the interval \([a, b] = [-8, 12]\). The cumulative distribution function of the Cauchy density is

\[
F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{x - \mu}{\alpha} \right),
\]

which in turn leads to the error bound \( 1 - 2\tan^{-1} \frac{10}{\pi} \approx 0.01 \). Figure 3.4 illustrates the quality of this approximation using 10 frequencies.

While some erratic behaviour is evident in the tails of the approximating density, nevertheless the quality of the approximation is remarkably good considering the small number of frequencies in use. The explanation for this surprising result is largely due to the fact that the maximum absolute error of \( F(a) + 1 - F(b) \approx 0.01 \) in the Fourier coefficients overestimates the true size of the error arising from the truncation of the integral defining the characteristic function. In reality the errors is less than \( F(a) + 1 - F(b) \) because of cancellations due to the fact that the integrand is the product of a slowly varying tail density (in this case the Cauchy density) and a rapidly oscillating function (in this case trigonometric functions represented in exponential form).

4. Pricing European Options. The successfully pricing of European option contracts for affine models of stochastic volatility requires knowledge of the marginal density of the asset price under the risk neutral measure. The difficulty, however, is that no closed-form expression for this density is available for even the simplest of the multivariate affine models used in finance, although it is well known that such models have characteristic function of generic form

\[
\chi(t, \omega) = \exp \left[ \beta_0(\tau, \omega) + \beta_1(\tau, \omega) Y + \sum_{k=2}^{M} \beta_k(\tau, \omega) X_k \right],
\]

\( \text{(4.1)} \)
where \( T \) is the maturity of the option, \( \tau = T - t \) is the backward variable and \( \omega \) is the characteristic variable associated with the non-dimensional variable \( Y \), here defined to be the logarithm of the ratio of asset price to strike price. The state variables \( \mathbf{X} = (X_2, \ldots, X_M) \) in expression (4.1) denote the values of the latent states of the system at time \( t \). Typically the functions \( \beta_0, \ldots, \beta_M \) are the solution of a system of \( (M + 1) \) ordinary differential equations with initial conditions \( \beta_1(0, \omega) = i\omega \) and \( \beta_0(0, \omega) = \beta_2(0, \omega), \ldots, \beta_M(0, \omega) = 0 \). In overview, there is a well trodden procedure that starts with the specification of the multivariate affine model and ends with the construction of \( \chi(t, \omega) \).

### 4.1. European call option.

In the case of a European call option with strike price \( K \) and maturity \( T \) on an asset with spot price \( S_0 \), the price of the option is

\[
e^{-rT} \int_K^\infty (S - K) \tilde{f}_Q(S_0, X_2 \cdots X_M, T \mid S; \theta) \, dS,
\]

where \( \tilde{f}_Q(S_0, X_2 \cdots X_M, T \mid S; \theta) \) is the marginal density of the asset price at the maturity of the option when \( S_0 \) is the spot price of the asset and \( (X_2, \ldots, X_M) \) are the spot values of the latent states. When expressed in terms of \( y = \log(S/K) \), the price of the option becomes

\[
C = e^{-rT}K \int_0^\infty (e^y - 1) f_Q(\xi, X_2 \cdots X_M, T \mid y; \theta) \, dy,
\]

where \( \xi = \log(S_0/K) \). The value of this integral is now approximated under the assumption that \( f_Q(\xi, X_2 \cdots X_M, T \mid y; \theta) \) is well approximated by a function of compact support over the interval \([a, b]\). Of course, the specific form taken by this approximation will depend on the choice of expression (a), (b) or (c) in equations (2.1), but in each case the approximation of \( f_Q(\xi, X_2 \cdots X_M, T \mid y; \theta) \) by a function of compact support in \([a, b]\) necessarily changes the interval of integration in expression (4.2) from \([0, \infty)\) to \([0, b]\). Thereafter it is straightforward Calculus to show that the cost of a call option on the basis of approximations (a) and (b) is

(a) \[ C = e^{-rT}K \left[ \frac{a_0}{2} (e^b - b - 1) + \sum_{k=1}^\infty a_k \frac{\omega_k e^b(-1)^k - \omega_k \cos \omega_k a - \sin \omega_k a}{\omega_k (1 + \omega_k^2)} \right], \]

(b) \[ C = e^{-rT}K \left[ \sum_{k=1}^\infty \frac{b_k}{\omega_k} \left( (e^b - 1)(-1)^{k-1} + \omega_k \sin \omega_k a - \cos \omega_k a + e^b(-1)^k \right) \right], \]

where \( \omega_k = k\pi/(b - a) \). Based on approximation (c) the price of a call option is

(c) \[ C = e^{-rT}K \left[ \frac{a_0}{2} (e^b - b - 1) + \sum_{k=1}^\infty a_k \frac{\lambda_k e^b - \lambda_k \cos \lambda_k a - \sin \lambda_k a}{\lambda_k (1 + \lambda_k^2)} + \frac{b_k}{\lambda_k} \left( 1 - e^b + \lambda_k \sin \lambda_k a - \cos \lambda_k a + e^b \right) \right], \]

where \( \lambda_k = 2k\pi/(b - a) \). The primary computational load in the computation of expressions (a), (b) and (c) resides in the calculation of trigonometric functions, and consequently individual terms of expression (c) require more arithmetical effort than the equivalent terms of either expression (a) or (b). However this difference in computational load is insignificant when compared with the more rapid convergence of
summation (c) compared with that of summations (a) and (b) due entirely to the fact that the frequencies used in approximation (c) are exactly double those used in approximations (a) and (b).

4.2. Digital option. The price of a digital option with strike price $K$ and maturity $T$ on an asset with spot price $S_0$ is

$$e^{-rT} \int_K^\infty \tilde{f}_Q(S_0, X_2 \cdots X_M, T \mid S; \theta) \, dS,$$

where $\tilde{f}_Q(S_0, X_2 \cdots X_M, T \mid S; \theta)$ is the marginal density of the asset price at the maturity of the option when the asset has spot price $S_0$ and $(X_2, \cdots, X_M)$ are the spot values of the latent states. When expressed in terms of the $y = \log(S/K)$, the price of the digital becomes

$$D = e^{-rT} \int_0^\infty f_Q(\xi, X_2 \cdots X_M, T \mid y; \theta) \, dy,$$  \hspace{1cm} (4.5)$$

where $\xi = \log(S_0/K)$. The value of this integral is now approximated under the assumption that $f_Q(\xi, X_2 \cdots X_M, T \mid y; \theta)$ is well represented by the procedures proposed in Section 2. Specifically, the price of a digital option computed from the half-range cosine, half-range sine and full-range Fourier series are respectively

(a) \hspace{1cm} D = e^{-rT} \left[ a_0 \frac{b}{2} + \sum_{k=1}^\infty a_k \frac{\sin \omega_k a}{\omega_k} \right], \\
(b) \hspace{1cm} D = e^{-rT} \sum_{k=1}^\infty b_k \left( \cos \omega_k a - (-1)^k \right), \hspace{1cm} (4.6)$$

where $\omega_k = k\pi/(b - a)$. Based on approximation (c) the price of a digital option is

(c) \hspace{1cm} D = e^{-rT} \left[ a_0 b + \sum_{k=1}^\infty a_k \frac{\sin \lambda_k a}{\lambda_k} + \frac{b_k}{\lambda_k} \left( \cos \lambda_k a - 1 \right) \right], \hspace{1cm} (4.7)$$

where $\lambda_k = 2k\pi/(b - a)$. As is the case with the pricing of a call option, the frequencies used in approximation (c) are double those used in approximations (a) and (b) suggesting that calculations based on expression (c) can be expected to converge more rapidly than those based on expressions (a) and (b).


$$\begin{align*}
    dY &= (r - V/2) \, dt + \sqrt{V} \left( \sqrt{1 - \rho^2} \, dW_1 + \rho \, dW_2 \right), \\
    dV &= \kappa_Q (\gamma_Q - V) \, dt + \sigma \sqrt{V} \, dW_2.
\end{align*}$$  \hspace{1cm} (5.1)$$

where $Y = \log S/K$, $V$ is the diffusion of asset price, $r$ is the risk-free rate of interest\(^3\), $\kappa_Q$ is the risk-neutral rate of mean reversion of volatility to the risk-neutral long run value of $\gamma_Q$, $\sigma$ scales the volatility of diffusion, $\rho$ is the local correlation between

\(^3\)The risk-free rate of interest may be adjusted to take account of dividend earnings, but this complication is ignored in order to retain simplicity in the analysis to follow.
returns and volatility and $dW_1$, $dW_2$ are increments in the independent Brownian motions $W_1$ and $W_2$. The parameter scaling the volatility premium is given by the difference between $\kappa_Q$ and its risk-averse value $\kappa_P$.

Suppose that $f(Y,V,t \mid Y = y, V = v, t = T)$ is the probability density function of equations (5.1) expressed in terms of the backward state $(Y,V)$ and the forward state $(y,v)$, then the backward Kolmogorov equation satisfied by the transitional probability density function of equations (5.1) is

$$\frac{\partial f}{\partial t} + (r - V/2) \frac{\partial f}{\partial Y} + \kappa_Q (\gamma_Q - V) \frac{\partial f}{\partial V} + \frac{V}{2} \left( \frac{\partial^2 f}{\partial Y^2} + 2\rho \sigma \frac{\partial^2 f}{\partial Y \partial V} + \sigma^2 \frac{\partial^2 f}{\partial V^2} \right) = 0. \quad (5.2)$$

Let $\chi(Y,V,t,\omega_Y,\omega_V)$ be the characteristic function of $f(Y,V,t \mid Y = y, V = v, t = T)$ with respect to the forward variables, that is,

$$\chi(Y,V,t,\omega_Y,\omega_V) = \int_{\mathbb{R}^2} f(Y,V,t \mid Y = y, V = v, t = T) e^{i(\omega_Y y + \omega_V v)} dy dv. \quad (5.3)$$

By taking the Fourier transform of equation (5.2) with respect to the backward variables, the function $\chi(Y,V,t,\omega_Y,\omega_V)$ is seen to satisfy the partial differential equation

$$\frac{\partial \chi}{\partial t} + (r - V/2) \frac{\partial \chi}{\partial Y} + \kappa_Q (\gamma_Q - V) \frac{\partial \chi}{\partial V} + \frac{V}{2} \left( \frac{\partial^2 \chi}{\partial Y^2} + 2\rho \sigma \frac{\partial^2 \chi}{\partial Y \partial V} + \sigma^2 \frac{\partial^2 \chi}{\partial V^2} \right) = 0. \quad (5.4)$$

with terminal condition $\chi(Y,V,T,\omega_Y,\omega_V) = \exp \left[ i(\omega_Y Y + \omega_V V) \right]$. Thereafter, it is straightforward to show that the anzatz

$$\chi(Y,V,t,\omega_Y,\omega_V) = \exp \left[ \beta_0(\tau) + \beta_1(\tau)Y + \beta_2(\tau)V \right], \quad \tau = T - t, \quad (5.5)$$

is a solution of equation (5.4) provided the coefficient functions $\beta_0(\tau)$, $\beta_1(\tau)$ and $\beta_2(\tau)$ satisfy the ordinary differential equations

$$\frac{d\beta_0}{d\tau} = r\beta_1 + \kappa_Q \gamma_Q \beta_2, \quad \frac{d\beta_1}{d\tau} = 0, \quad \frac{d\beta_2}{d\tau} = -\frac{\beta_1}{2} - \kappa_Q \beta_2 + \frac{1}{2} \left( \beta_1^2 + 2\rho \sigma \beta_1 \beta_2 + \sigma^2 \beta_2^2 \right). \quad (5.6)$$

The characteristic function of the marginal density of the terminal value of $y = \log(S_T/K)$ requires the solution of equations (5.6) with initial conditions $\beta_0(0) = 0$, $\beta_1(0) = i\omega_V$ and $\beta_2(0) = 0$ for the particular values of $\omega_V$ needed to construct the half-range and full-range Fourier series approximations of transitional probability density.

On a practical note, the fact that the characteristic function of $\log(S/K)$ embeds the risk-free rate of interest suggest at first sight that the coefficients $\beta_0(t), \cdots \beta_M(t)$ must be computed whenever the risk-free rate changes, potentially each day, and not just when the parameters of the model are changed. The key to avoiding this difficulty is to note that the risk-free rate of interest enters the calculation of $\beta_0$ alone, and that the behaviours of $\beta_1, \cdots, \beta_M$ are independent of the behaviour of $\beta_0$. Consequently, the resolution of this practical dilemma is to divide the calculation of $\beta_0$ into two stages with the first stage treating only those calculations that involve the risk-free rate and the second stage treating everything that is independent of the value of the
risk-free rate. Each day both stages are brought together in the calculation of the expected payoff, but only the first stage of calculation need be done on a day-to-day basis. In particular, this calculation is very easy as it amounts to adding $rT\omega_n$ to the second stage calculation of $\beta_0$ in the computation of $\chi(\omega_n)$.

6. Error analysis. Let $f_Q(y)$ denote the marginal density of $Y = \log(S/K)$, where $K$ is the strike price of an asset. The purpose of this section is to demonstrate that the error in estimating the price of a call option using only that part of the density $f_Q(y)$ in the finite interval $[a, b]$ can be made arbitrarily small by choosing a suitably large interval. The function $f_Q(y)$, when truncated to the interval $[a, b]$, is assumed to satisfy the Dirichlet conditions and therefore is guaranteed to have a convergent Fourier series on $[a, b]$.

The approximation procedure introduces three different errors, the first of which is the error in the true price of a call option due to the loss of contributions from values of $y$ exterior to $[a, b]$. With this approximation in place, the price of the call option is approximated by the expression

$$e^{-rT}K \int_a^b (e^y - 1) f_Q(y) \, dy. \quad (6.1)$$

In particular, a straightforward analysis indicates that the error in replacing the true cost of the call option by expression (6.1) is

$$e^{-rT}K \int_0^b (e^y - 1) f_Q(y) \, dy - e^{-rT}K \int_0^a (e^y - 1) f_Q(y) \, dy$$

$$= e^{-rT}K \left[ (e^b - 1) (1 - F_Q(b)) + \int_b^\infty e^y (1 - F_Q(y)) \, dy \right], \quad (6.2)$$

where $F_Q(y)$ is the cumulative function of $y$. Evidently the error in pricing a call option by truncating the marginal density of $Y$ at $y = b$ is the sum of two positive contributions, both of which are driven by the extent to which the restriction of the marginal density of $Y$ to the interval $[a, b]$ fails to capture density in the upper tail of the distribution of $Y$. Thus expression (6.1) always underestimates the true value of the option. Clearly this error can be made arbitrarily small by taking $[a, b]$ to be a suitably large interval.

Moreover, at a casual glance it would appear that this error is independent of the approximate representation chosen for $f_Q(y)$ in $[a, b]$. This, however, is not strictly true; the total density captured by $f_Q(y)$ in $[a, b]$ is less than one, and so representations of $f_Q(y)$ that place unit density in $[a, b]$ will compensate for some of the pricing error identified through the term $e^{-rT}K (e^b - 1) (1 - F_Q(b))$ of equation (6.2). The assignment of unit density to $[a, b]$ is a property of the half-range Fourier cosine series and the full-range Fourier series approximations but not shared by the half-range Fourier sine series approximation.

The strategy is now to replace the transitional density $f_Q(y)$ in the interval $[a, b]$ by the Fourier series

$$f_Q(y) = \frac{a_0}{2} + \sum_{k=1}^\infty a_k \cos \omega_k (y - a) + b_k \sin \omega_k (y - a), \quad (6.3)$$

where the choice of frequencies $\omega_k$ and the values of $a_k$, $b_k$ will depend on the choice of Fourier representation. The two remaining errors pertain to the calculation of
expression (6.1). The first error arises from the misspecification of the coefficients $a_k, b_k$ in equation (6.3) by the respective coefficients $A_k, B_k$ computed directly from the characteristic function $\chi(\omega_k)$, of $f_Q(y)$ by the formula

$$A_k + iB_k = \frac{e^{-i\omega_k a}}{b - a} \chi(\omega_k).$$

(6.4)

The second error is due to the truncation of the Fourier series (6.3) at a finite number of terms, say $N$ terms. Therefore the price of the option based on this approximation strategy is

$$C = e^{-rT} K \left[ \frac{A_0}{2} \int_0^b (e^y - 1) \, dy + \sum_{k=1}^N A_k \int_0^b (e^y - 1) \cos \omega_k (y - a) \, dy 
+ \sum_{k=1}^N B_k \int_0^b (e^y - 1) \sin \omega_k (y - a) \, dy \right].$$

(6.5)

in which $f_Q(y)$ in expression (6.1) has been replaced by the first $(N+1)$ terms of the Fourier series (6.3) with misspecified coefficients $A_k$ and $B_k$. The error introduced by this approximation is

$$e^{-rT} K \int_0^b (e^y - 1) f_Q(y) \, dy - C$$

(6.6)

which has explicit expression

$$e^{-rT} K \left[ \frac{(a_0 - A_0)}{2} \int_0^b (e^y - 1) \, dy + \sum_{k=1}^N (a_k - A_k) \int_0^b (e^y - 1) \cos \omega_k (y - a) \, dy 
+ \sum_{k=1}^N (b_k - B_k) \int_0^b (e^y - 1) \sin \omega_k (y - a) \, dy 
+ \sum_{k=N+1}^{\infty} a_k \int_0^b (e^y - 1) \cos \omega_k (y - a) \, dy + b_k \int_0^b (e^y - 1) \sin \omega_k (y - a) \, dy \right].$$

(6.7)

The misspecification error in the coefficients $a_k, b_k$ due to the use of the coefficients $A_k, B_k$ is determined from the identity

$$a_k + ib_k = \frac{1}{b - a} \int_a^b f_Q(y) e^{i\omega_k (y - a)} \, dy = A_k + iB_k - \frac{e^{-i\omega_k a}}{b - a} \left[ \int_{-\infty}^a f_Q(y) e^{i\omega_k y} \, dy + \int_b^\infty f_Q(y) e^{i\omega_k y} \, dy \right].$$

Standard properties of integral Calculus guarantee that

$$\left| (a_k + ib_k) - (A_k + iB_k) \right| = \frac{1}{b - a} \left| \int_{-\infty}^a f_Q(y) e^{i\omega_k y} \, dy + \int_b^\infty f_Q(y) e^{i\omega_k y} \, dy \right|$$

$$\leq \varepsilon_1 = \frac{1}{b - a} \int_{-\infty}^a f_Q(y) \, dy + \frac{1}{b - a} \int_b^\infty f_Q(y) \, dy,$$

from which it follows directly that

$$|a_k - A_k| \leq \varepsilon_1, \quad |b_k - B_k| \leq \varepsilon_1,$$

(6.8)
for all values of \( k \), where
\[
\varepsilon_1 = \frac{1}{b - a} \left[ F_Q(a) + 1 - F_Q(b) \right]. \tag{6.9}
\]

Thus the misspecification error in the Fourier coefficients is again driven entirely by the choice of interval \([a, b]\) via the magnitude of the cumulative function \( F_Q(y) \) at \( y = a \) and \( y = b \). In practice this error will be significantly smaller than the maximum bound given in equation (6.8) as a result of arithmetical cancellation due to the oscillatory nature of the integrand.

In conclusion, the total error in pricing a call option, namely
\[
\text{Error} = e^{-rT} K \int_0^\infty (e^y - 1) f_Q(y) \, dy - C, \tag{6.10}
\]
is constructed by connecting together equation (6.2) for the error arising in the truncation of the density \( f_Q(y) \) to the finite interval \([a, b]\), equation (6.5) for the value of the option in terms of the misspecified Fourier coefficients \( A_k, B_k \), and equation (6.7) for the size of the resulting misspecification error in the option price. The result is that
\[
\text{Error} = e^{-rT} K \left[ (e^b - 1)(1 - F_Q(b)) + \int_b^\infty e^y (1 - F_Q(y)) \, dy \right] \\
+ e^{-rT} K \left[ \frac{(a_0 - A_0)}{2} \int_0^b (e^y - 1) \, dy \right. \\
+ \sum_{k=1}^N (a_k - A_k) \int_0^b (e^y - 1) \cos \omega_k (y - a) \, dy \right. \\
+ \sum_{k=1}^N (b_k - B_k) \int_0^b (e^y - 1) \sin \omega_k (y - a) \, dy \left. \right. \\
+ \sum_{k=N+1}^\infty a_k \int_0^b (e^y - 1) \cos \omega_k (y - a) \, dy \\
+ \sum_{k=N+1}^\infty b_k \int_0^b (e^y - 1) \sin \omega_k (y - a) \, dy \right]. \tag{6.11}
\]

Each integral is replaced by its value and the triangle inequality is used to deduce that
\[
|\text{Error}| \leq e^{-rT} K \left[ (e^b - 1)(1 - F_Q(b)) + \int_b^\infty e^y (1 - F_Q(y)) \, dy \right. \\
+ \varepsilon_1 \left( e^b - b - 1 \right) + \varepsilon_1 \sum_{k=1}^N \left| \sin \omega_k a - \omega_k \left( e^b \cos \omega_k (b - a) - \cos \omega_k a \right) \right| \\
+ \varepsilon_1 \sum_{k=1}^N \left| \frac{(e^b \omega_k^2 - 1) \cos \omega_k (b - a) + \cos \omega_k a - \omega_k \sin \omega_k a}{\omega_k (1 + \omega_k^2)} \right| \\
+ \sum_{k=N+1}^\infty \left| a_k \right| \left| \sin \omega_k a - \omega_k \left( e^b \cos \omega_k (b - a) - \cos \omega_k a \right) \right| \\
+ \sum_{k=N+1}^\infty \left| b_k \right| \left| \frac{(e^b \omega_k^2 - 1) \cos \omega_k (b - a) + \cos \omega_k a - \omega_k \sin \omega_k a}{\omega_k (1 + \omega_k^2)} \right|. \tag{6.12}
\]
The contribution to the error from the first, second, third and fourth terms on the right hand side of inequality (6.12) is dominated by the behaviour of \( F_Q(a) \) and \( 1 - F_Q(b) \). By choosing the interval \([a, b]\) suitably large \( F_Q(a) \) and \( 1 - F_Q(b) \) can be made arbitrarily small. The behaviour of the fifth and sixth terms on the right hand side of inequality (6.12) depends on the choice of \( N \). Bearing in mind that \( \omega_k = O(k) \), then it clear that there are positive constants \( C_1 \) and \( C_2 \) such that

\[
\left| \frac{\sin \omega_k a - \omega_k \left( e^b \cos \omega_k (b - a) - \cos \omega_k a \right)}{\omega_k (1 + \omega_k^2)} \right| \leq \frac{e^b + 1}{\omega_k \sqrt{1 + \omega_k^2}} \leq \frac{C_1}{k^2},
\]

\[
\left| \frac{(e^{b \omega_k^2} - 1) \cos \omega_k (b - a) + \cos \omega_k a - \omega_k \sin \omega_k a}{\omega_k (1 + \omega_k^2)} \right| \leq \frac{e^b \sqrt{1 + \omega_k^2} + 1}{\omega_k \sqrt{1 + \omega_k^2}} \leq \frac{C_2}{k}.
\]

The well known result that if \( f_Q(y) \) is a continuous function\(^4\) of \( y \) then \( |a_k| < K/k \) and \( |b_k| < K/k \) leads to the conclusion that the component terms of the fourth and fifth series of inequality (6.12) are \( O(k^{-3}) \) and \( O(k^{-2}) \) respectively. Thus it is straightforward to show that the fifth and sixth terms on the right hand side of inequality (6.12) can also be made arbitrarily small by a suitably large choice of \( N \).

In conclusion, the error in pricing a call option can be made arbitrarily small by restricting the marginal probability density to a finite interval \([a, b]\) and approximating the density in that interval by a misspecified Fourier series constructed from the characteristic function at the appropriate frequencies.

7. Performance under simulation. A series of simulation experiments was undertaken in order to examine the efficacy of the half-range Fourier cosine series, half-range Fourier sine series and full-range Fourier series in respect of how accurately these approximations price European Call options and Digital options. The first experiment prices options in a Black-Scholes world so that a closed-form solution may be used to assess the pricing error, whereas the second experiment prices the same options using Heston’s model of stochastic volatility.

7.1. Black-Scholes pricing. Assume that the asset price, \( S \), follows a geometric Brownian motion

\[
dS = S(r \, dt + \sigma \, dW)
\]

in which \( dW \) is the increment of a Wiener process. The advantage of using this specification is that, following the seminal work of Black and Scholes [1], exact prices are known for each type of option for all combinations of spot price, \( S_0 \), strike price, \( K \), and maturity, \( T \). It follows, therefore, that the relative percentage pricing error incurred using numerical methods based on Fourier series for a variety of different strike prices and maturities can be identified exactly.

Three major experiments are performed. In each of these experiments \( r = 0.05 \), \( \sigma = 0.15 \) per year, \( S_0 = 1000 \) and the strikes \( K \) are taken to be 1200, 1000 and 800 respectively so as to examine the performance of the algorithms when the options are out-of-the-money, at-the-money and in-the-money. In each simulation the accuracy of the various approximations of transitional density is assessed for prescribed values

---

\(^4\)Sharper results can be obtained if \( f_Q(y) \in C^p[a, b] \) for \( p > 1 \). However the function represented by the half-range Fourier cosine series is not generally differentiable at \( x = a \), and so the convergence argument is based on the weakest condition satisfied by Fourier coefficients.
of $a$ and $b$ in which the size of the interval $[a, b]$ is expressed as a multiple of $\sigma \sqrt{T}$ ranging from 5 to 9. The results of these exercises are reported in Tables 7.1, 7.2 and 7.3 with each table reporting the results for maturities of one month (panel (a)), three months (panel (b)) and six months (panel (c)).

<table>
<thead>
<tr>
<th>(a) Factor</th>
<th>Call Strike = 1200 T = 0.083</th>
<th>Digital Strike = 1200 T = 0.083</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cos</td>
<td>Sin</td>
<td>Full</td>
</tr>
<tr>
<td>5.00</td>
<td>-4.160746</td>
<td>-17.315659 -7.719366</td>
</tr>
<tr>
<td>6.00</td>
<td>-0.000537</td>
<td>-0.129887 -0.052720</td>
</tr>
<tr>
<td>7.00</td>
<td>0.008273</td>
<td>0.018872 0.001766</td>
</tr>
<tr>
<td>8.00</td>
<td>0.058873</td>
<td>0.019228 0.002259</td>
</tr>
<tr>
<td>9.00</td>
<td>0.015106</td>
<td>0.018981 0.002546</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(b) Factor</th>
<th>Call Strike = 1200 T = 0.083</th>
<th>Digital Strike = 1200 T = 0.083</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cos</td>
<td>Sin</td>
<td>Full</td>
</tr>
<tr>
<td>5.00</td>
<td>-0.006890</td>
<td>-0.084497 -0.035497</td>
</tr>
<tr>
<td>6.00</td>
<td>-0.000014</td>
<td>-0.000471 -0.000204</td>
</tr>
<tr>
<td>7.00</td>
<td>0.000000</td>
<td>0.000004 -0.000001</td>
</tr>
<tr>
<td>8.00</td>
<td>0.000000</td>
<td>0.000005 -0.000001</td>
</tr>
<tr>
<td>9.00</td>
<td>0.000000</td>
<td>0.000005 -0.000001</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(c) Factor</th>
<th>Call Strike = 1200 T = 0.500</th>
<th>Digital Strike = 1200 T = 0.500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cos</td>
<td>Sin</td>
<td>Full</td>
</tr>
<tr>
<td>5.00</td>
<td>-0.001412</td>
<td>-0.020338 -0.009446</td>
</tr>
<tr>
<td>6.00</td>
<td>-0.000005</td>
<td>-0.000115 -0.000055</td>
</tr>
<tr>
<td>7.00</td>
<td>0.000000</td>
<td>0.000000 -0.000000</td>
</tr>
<tr>
<td>8.00</td>
<td>0.000000</td>
<td>0.000000 -0.000000</td>
</tr>
<tr>
<td>9.00</td>
<td>0.000000</td>
<td>0.000000 -0.000000</td>
</tr>
</tbody>
</table>

Table 7.1

Percentage errors recorded by the half-range cosine series, half-range sine series and full-range Fourier series of pricing a European call option and a digital option with strike $K = 1200$ and maturities one month (panel (a)), three months (panel (b)) and six months (panel (c)). The factor refers to the multiple of $\sigma \sqrt{T}$ used to establish the interval over which to compute the numerical approximation.

Two very clear general conclusions emerge from these results.

1. For options that are deep in-the-money (Table 7.3) in which $S_0 = 1000$ and $K = 800$, or at-the-money (Table 7.2) in which $S_0 = 1000$ and $K = 1000$, it matters little (if at all) which of these three pricing algorithms are chosen irrespective of the maturity of the option. Indeed the relative pricing error is zero for all practical purposes.

2. For options that are deep out-of-the-money (Table 7.1) in which $S_0 = 1000$ and $K = 1200$, the choice of algorithm is more important. The results may be summarized succinctly as follows.

(a) The half-range Fourier sine series does not perform as well as the other
two approximations and its use is therefore not recommended.

(b) The half-range Fourier cosine series and the full-range Fourier series both perform relatively well. When the size of the interval of approximation is taken to be a relatively small multiple of \( \sigma \sqrt{T} \), namely either 5 or 6, then the half-range Fourier cosine series performs better that the full-range Fourier series. As the multiple of \( \sigma \sqrt{T} \) increases and the size of the interval of approximation becomes larger, the full-range Fourier series begins to dominate. When the interval of approximation has size \( 10 \times \sigma \sqrt{T} \), then the full-range Fourier series is unambiguously superior to the half-range Fourier cosine series, particularly for options of short maturity.

On the basis of this analysis and on accuracy grounds, it is hard to ignore the claim that the full-range Fourier series is the algorithm of choice when using Fourier methods to price options. Moreover, the full-range Fourier series converges faster than either

---

### Table 7.2

Percentage errors recorded by the half-range cosine series, half-range sine series and full-range Fourier series of pricing a European call option and a digital option with strike \( K = 1000 \) and maturities one month (panel (a)), three months (panel (b)) and six months (panel (c)). The factor refers to the multiple of \( \sigma \sqrt{T} \) used to establish the interval over which to compute the numerical approximation.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Call Strike = 1000 T = 0.083</th>
<th>Digital Strike = 1000 T = 0.083</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cos</td>
<td>Sin</td>
</tr>
<tr>
<td>5.00</td>
<td>0.000044</td>
<td>0.000313</td>
</tr>
<tr>
<td>6.00</td>
<td>0.000000</td>
<td>0.000002</td>
</tr>
<tr>
<td>7.00</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>8.00</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>9.00</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Factor</th>
<th>Call Strike = 1000 T = 0.250</th>
<th>Digital Strike = 1000 T = 0.250</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cos</td>
<td>Sin</td>
</tr>
<tr>
<td>5.00</td>
<td>0.000063</td>
<td>0.001375</td>
</tr>
<tr>
<td>6.00</td>
<td>0.000000</td>
<td>0.000007</td>
</tr>
<tr>
<td>7.00</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>8.00</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>9.00</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Factor</th>
<th>Call Strike = 1000 T = 0.500</th>
<th>Digital Strike = 1000 T = 0.500</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cos</td>
<td>Sin</td>
</tr>
<tr>
<td>5.00</td>
<td>0.000116</td>
<td>0.002208</td>
</tr>
<tr>
<td>6.00</td>
<td>0.000000</td>
<td>0.000012</td>
</tr>
<tr>
<td>7.00</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>8.00</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>9.00</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
the half-range sine or cosine series and is therefore likely to price options more rapidly. These themes are explored in more detail in the pricing based on Heston’s model of stochastic volatility.

7.2. Heston’s Model. A total of approximately 40,000 options over ten years were generated by simulation of Heston’s model. Sixteen options were generated each day spread over 4 maturities ranging from 92 to 5 days and 4 strikes two of which are initialised at 20% out of and into the money and two of which are initialised at 7% out of and into the money. The half-range Fourier cosine series and the full-range Fourier series are then used to price call options and digital options with these strikes. In this instance no exact solutions are available, and so the accuracy of each method in respect of each type of option is gauged by comparison against values calculated using a large interval. The left hand and middle columns of Table 7.4 show respectively the $L_2$ and $L_1$ relative pricing errors for out-of-the-money call options calculated using the half-range cosine series and the full-range Fourier series. The right hand column

<table>
<thead>
<tr>
<th>(a) Factor</th>
<th>Call Strike = 800 T = 0.083</th>
<th>Digital Strike = 800 T = 0.083</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cos</td>
<td>-0.000003 -0.000184 -0.000049</td>
<td>-0.00043 -0.000127 -0.000068</td>
</tr>
<tr>
<td>Sin</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cos</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sin</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(b) Factor</th>
<th>Call Strike = 800 T = 0.250</th>
<th>Digital Strike = 800 T = 0.250</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cos</td>
<td>-0.000011 -0.00033 -0.00133</td>
<td>-0.000000 -0.000111 -0.000041</td>
</tr>
<tr>
<td>Sin</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cos</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sin</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(c) Factor</th>
<th>Call Strike = 800 T = 0.500</th>
<th>Digital Strike = 800 T = 0.500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cos</td>
<td>-0.000022 -0.00058 -0.000259</td>
<td>-0.000008 -0.000147 -0.000063</td>
</tr>
<tr>
<td>Sin</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cos</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sin</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7.3
Percentage errors recorded by the half-range cosine series, half-range sine series and full-range Fourier series of pricing a European call option and a digital option with strike $K = 800$ and maturities one month (panel (a)), three months (panel (b)) and six months (panel (c)). The factor refers to the multiple of $\sigma \sqrt{T}$ used to establish the interval over which to compute the numerical approximation.
of Table 7.4 shows the CPU time (secs) needed to perform the calculations. The left hand column specifies the length of the interval $[a,b]$ in multiples of $\sigma$, the expected standard deviation of the distribution of $Y$ about $\log(S_0/K)$.

<table>
<thead>
<tr>
<th>$(b - a)/\sigma$</th>
<th>Out-of-the-money call $L_2$ Error</th>
<th>Out-of-the-money call $L_1$ Error</th>
<th>Timings (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cos Series</td>
<td>Full Series</td>
<td>Cosine</td>
</tr>
<tr>
<td>10</td>
<td>0.000228</td>
<td>0.172011</td>
<td>0.000289</td>
</tr>
<tr>
<td>12</td>
<td>0.000523</td>
<td>0.056627</td>
<td>0.000045</td>
</tr>
<tr>
<td>14</td>
<td>0.000053</td>
<td>0.016726</td>
<td>0.000029</td>
</tr>
<tr>
<td>16</td>
<td>0.000242</td>
<td>0.004654</td>
<td>0.000030</td>
</tr>
<tr>
<td>18</td>
<td>0.000374</td>
<td>0.001251</td>
<td>0.000030</td>
</tr>
<tr>
<td>20</td>
<td>0.000190</td>
<td>0.000330</td>
<td>0.000035</td>
</tr>
<tr>
<td>22</td>
<td>0.000064</td>
<td>0.000086</td>
<td>0.000028</td>
</tr>
<tr>
<td>24</td>
<td>0.000460</td>
<td>0.000023</td>
<td>0.000030</td>
</tr>
<tr>
<td>26</td>
<td>0.000058</td>
<td>0.000010</td>
<td>0.000027</td>
</tr>
<tr>
<td>28</td>
<td>0.000315</td>
<td>0.000008</td>
<td>0.000029</td>
</tr>
<tr>
<td>30</td>
<td>0.000219</td>
<td>0.000007</td>
<td>0.000035</td>
</tr>
</tbody>
</table>

**Table 7.4**

The table shows the $L_2$ and $L_1$ relative pricing errors calculated from the half-range cosine series and the full-range Fourier series for out-of-the-money call options based on Heston’s model. The timings refer to the times (seconds) needed to price 12 daily call options over 10 years with maturities ranging from 5 to 92 days.

A clear observation from Table 7.4 is that the half-range Fourier cosine series performs well in respect of the $L_2$ measure for shorter intervals and in the $L_1$ measure for almost all choices of interval length. On the other hand the performance of the full-range Fourier series is poor with regard to both measures for shorter intervals. However, the quality of approximation provided by the half-range cosine series is erratic as the size of the Fourier window increases whereas that provided by the full-range Fourier series improves systematically to the extent that its performance surpasses that of the half-range Fourier cosine series for intervals of length 24 standard deviations or more. Furthermore, this level of accuracy is achieved by the full-range Fourier series in approximately 25% less time than that required by the half-range Fourier cosine series.

Fang and Oosterlee [8] suggest choosing intervals of length 20 standard deviations. In this problem, it is evident that the half-range Fourier cosine series still enjoys an advantage over the full-range Fourier series for intervals of this length. The suggestion of this investigation is that 20 standard deviations should be regarded as a minimum length of interval. Table 7.5 shows the $L_2$ and $L_1$ relative pricing errors calculated from the half-range Fourier cosine series and the full-range Fourier series in respect of in-the-money call options based on Heston’s model.

The results reported in Table 7.5 exhibit a similar pattern of behaviour to those reported in Table 7.4, namely that for intervals of short length the half-range Fourier cosine series outperforms the full-range Fourier series with respect to both the $L_2$ and $L_1$ measures of accuracy, but that this dominance vanishes when the length of
the interval of approximation is 24 standard deviations or more. In particular, both approaches generate noticeably smaller relative errors for in-the-money options than for out-of-the-money options. Because in-the-money call options carry a higher price, one inference of this observed reduction in relative error is that the absolute pricing error in using each Fourier representation is insensitive to the moniness of the option. Tables 7.6 and 7.7 report the performance of the half-range Fourier cosine series and the full-range Fourier series when used to price digital options.

The results presented in Tables 7.6 and 7.7 demonstrate that both the half-range Fourier cosine series and the full-range Fourier series both generate good estimates of the price of digital options with the former performing better than the latter for intervals of short length, but with this advantage disappearing when the length of the interval is increased.

The results reported in Tables 7.4-7.7 are characterised by two common denominators. First, the computational time required by the half-range Fourier cosine series is uniformly longer than that required by the full-range Fourier series for an interval of fixed length. The simple explanation for this observation is that the full-range Fourier series uses larger frequencies for a given length of interval, and therefore the full-range Fourier series converges more rapidly. Second, the pricing of call options and digital options using the half-range Fourier cosine series representation of transitional density is noticeably better than the corresponding pricing using the full-range Fourier series for short intervals \((a,b)\), but this advantage vanishes in the case of the digital option when the interval becomes suitably large and is reversed in the case of a call option. The primary explanation for this behaviour lies in the realisation that tail density is characterised by long wavelengths, or equivalently by the presence of low frequency terms, whereas peaky density is characterised by short wavelengths.
The table shows the $L_2$ and $L_1$ relative pricing errors calculated from the half-range cosine series and the full-range Fourier series for out-of-the-money digital options based on Heston’s model. The timings refer to the times (seconds) needed to price 12 daily call options over 10 years with maturities ranging from 5 to 92 days.

<table>
<thead>
<tr>
<th>$(b - a)/\sigma$</th>
<th>Digital out-of-the-money $L_2$ Error</th>
<th>Digital out-of-the-money $L_1$ Error</th>
<th>Timings (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cos Series Full Series</td>
<td>Cos Series Full Series</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.000020 0.005846</td>
<td>0.000000 0.000328</td>
<td>0.172 0.094</td>
</tr>
<tr>
<td>12</td>
<td>0.000006 0.001753</td>
<td>0.000000 0.000081</td>
<td>0.203 0.125</td>
</tr>
<tr>
<td>14</td>
<td>0.000020 0.000504</td>
<td>0.000000 0.000020</td>
<td>0.234 0.125</td>
</tr>
<tr>
<td>16</td>
<td>0.000007 0.000141</td>
<td>0.000000 0.000005</td>
<td>0.265 0.141</td>
</tr>
<tr>
<td>18</td>
<td>0.000028 0.000047</td>
<td>0.000000 0.000001</td>
<td>0.281 0.156</td>
</tr>
<tr>
<td>20</td>
<td>0.000004 0.000022</td>
<td>0.000000 0.000000</td>
<td>0.312 0.172</td>
</tr>
<tr>
<td>22</td>
<td>0.000020 0.000020</td>
<td>0.000000 0.000000</td>
<td>0.343 0.187</td>
</tr>
<tr>
<td>26</td>
<td>0.000020 0.000020</td>
<td>0.000000 0.000000</td>
<td>0.390 0.203</td>
</tr>
<tr>
<td>30</td>
<td>0.000022 0.000020</td>
<td>0.000000 0.000000</td>
<td>0.437 0.250</td>
</tr>
</tbody>
</table>

or equivalently high frequency terms. For a given length of interval, the frequencies present in the half-range Fourier cosine expansion are smaller than the frequencies present in the full-range Fourier series, and therefore when the length of the interval $[a, b]$ is small, the half-range Fourier cosine series better captures tail behaviour. Increasing the length of the interval lowers the range of frequencies admitted to the full-range Fourier series expansion, which in turn improves its ability to handle tail density. However the same increase in length, although it allows lower frequencies to
enter the half-range Fourier cosine series representation of transitional density, their presence will have an impact only if there is significant tail density to be characterised by these new frequencies. On the other hand, the half-range Fourier cosine series has the drawback that the (periodic) function it represents is not differentiable at \( x = a \) in contrast to the behaviour of the function represented by the full-range Fourier series.

8. Conclusion. One clear conclusion from these calculations is that the half-range Fourier cosine series and the full-range Fourier series perform uniformly better than the half-range Fourier sine series. The half-range Fourier cosine series and the full-range Fourier series both perform with credit. When the length of the interval \([a, b]\) is relatively small, say ten or so standard deviations, it is clear that the half-range Fourier cosine series outperforms the full-range Fourier series over the same interval. On the other hand for intervals of larger length the full-range Fourier series is at least as good as the half-range Fourier cosine series, and outperforms the latter in pricing out-of-the-money call options, in particular, with maturities of three months or less. In fact the full-range Fourier series outperforms the half-range Fourier cosine series in all circumstances provided the interval \([a, b]\) is suitably large, although the effect is so small as not to be significant in practice. The explanation for this behaviour lies in the fact that the half-range Fourier cosine series, although not representing a differentiable function, nevertheless always uses a lower spectrum of frequencies than the full-range Fourier series and therefore enjoys an initial advantage in describing tail density. As the interval length is increased, this advantage vanishes. On the other hand the larger spectrum of the full-range Fourier series guarantees more rapid convergence. Of course, timing issues may not be important if small numbers of model call prices are to be calculated, but when each and every evaluation of the likelihood function requires in excess of one billion calculations of the model call price as occurs with a particle filtering algorithm, timing issues are now significant once adequate numerical accuracy is assured.

REFERENCES

