Solutions to Examples on
Stochastic Differential Equations

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Q 1. Let \( W_1(t) \) and \( W_2(t) \) be two Wiener processes with correlated increments \( \Delta W_1 \) and \( \Delta W_2 \) such that \( \mathbb{E}[\Delta W_1 \Delta W_2] = \rho \Delta t \). Prove that \( \mathbb{E}[W_1(t)W_2(t)] = \rho t \). What is the value of \( \mathbb{E}[W_1(t)W_2(s)] \)?

**Answer**

Let \( 0 = t_0 < t_1 < t_2 < \cdots < t_n = t \) be a dissection of the interval \( [0, t] \), then

\[
W^{(1)}_t = \sum_{k=1}^{n} \Delta W^{(1)}_k, \quad W^{(2)}_t = \sum_{k=1}^{n} \Delta W^{(2)}_k.
\]

where \( W^{(1)}_k \) and \( W^{(2)}_k \) are zero mean Gaussian deviates satisfying \( \mathbb{E}[W^{(1)}_k W^{(2)}_k] = \rho (t_k - t_{k-1}) \). Thus

\[
\mathbb{E}[W^{(1)}_t W^{(2)}_t] = \mathbb{E}\left[ \left( \sum_{k=1}^{n} \Delta W^{(1)}_k \right) \left( \sum_{j=1}^{n} \Delta W^{(2)}_j \right) \right] = \sum_{k,j=1}^{n} \mathbb{E}[\Delta W^{(1)}_k \Delta W^{(2)}_j]
\]

However, \( \Delta W^{(1)}_k \) and \( \Delta W^{(2)}_j \) are independent Gaussian deviates if \( k \neq j \) and so \( \mathbb{E}[\Delta W^{(1)}_k \Delta W^{(2)}_j] = \rho (t_k - t_{k-1}) \delta(k - j) \). Thus

\[
\mathbb{E}[W^{(1)}_t W^{(2)}_t] = \sum_{k=1}^{n} \rho (t_k - t_{k-1}) = \rho (t_n - t_0) = \rho t.
\]

Q 2. Let \( W(t) \) be a Wiener process and let \( \lambda \) be a positive constant. Show that \( \lambda^{-1} W(\lambda^2 t) \) and \( t W(1/t) \) are each Wiener processes.

**Answer**

This question concerns the properties of a random variable under changes of variable. We need to show that each random variable is Gaussian distributed with mean value zero and variance \( t \).

(a) Here \( t \) is a parameter and \( W(\lambda^2 t) \) is a Gaussian random variable with mean value zero and variance \( \lambda^2 t \). Let \( Y = \lambda^{-1} W(\lambda^2 t) \) then

\[
f_Y = f_W \frac{dW}{dY} = \lambda f_W = \lambda \frac{1}{\sqrt{2\pi\lambda^2 t}} \exp\left( -\frac{W^2}{2\lambda^2 t} \right) = \frac{1}{\sqrt{2\pi t}} \exp\left( -\frac{\lambda^2 Y^2}{2\lambda^2 t} \right) = \frac{1}{\sqrt{2\pi t}} \exp\left( -\frac{Y^2}{2t} \right).
\]

Thus \( Y \) is a Gaussian deviate with mean value zero and variance \( t \).

(b) Here \( t \) is again a parameter and \( W(1/t) \) is a Gaussian random variable with mean value zero and variance \( 1/t \). Let \( Y = t W(1/t) \) then

\[
f_Y = f_W \frac{dW}{dY} = \frac{1}{t} f_W = \frac{1}{t} \frac{1}{\sqrt{2\pi(1/t)}} \exp\left( -\frac{W^2}{2(1/t)} \right) = \frac{1}{\sqrt{2\pi t}} \exp\left( -\frac{(Y/t)^2}{2\lambda^2(1/t)} \right) = \frac{1}{\sqrt{2\pi t}} \exp\left( -\frac{Y^2}{2t} \right).
\]

Thus \( Y \) is a Gaussian deviate with mean value zero and variance \( t \).
Q 3. Suppose that \((\varepsilon_1, \varepsilon_2)\) is a pair of uncorrelated \(N(0, 1)\) deviates.

(a) By recognising that \(\xi_x = \sigma_x \varepsilon_1\) has mean value zero and variance \(\sigma_x^2\), construct a second deviate \(\xi_y\) with mean value zero such that the Gaussian deviate \(X = [\xi_x, \xi_y]^T\) has mean value zero and correlation tensor

\[
\Omega = \begin{bmatrix}
\sigma_x^2 & \rho \sigma_x \sigma_y \\
\rho \sigma_x \sigma_y & \sigma_y^2
\end{bmatrix}
\]

where \(\sigma_x > 0, \sigma_y > 0\) and \(|\rho| < 1\).

(b) Another possible way to approach this problem is to recognise that every correlation tensor is similar to a diagonal matrix with positive entries. Let

\[
\alpha = \frac{\sigma_x^2 - \sigma_y^2}{2}, \quad \beta = \frac{1}{2} \sqrt{(\sigma_x^2 + \sigma_y^2)^2 - 4(1 - \rho^2)\sigma_x^2 \sigma_y^2} = \sqrt{\alpha^2 - \rho^2 \sigma_x^2 \sigma_y^2}.
\]

Show that

\[
Q = \frac{1}{\sqrt{2\beta}} \begin{bmatrix}
\sqrt{\beta + \alpha} & -\sqrt{\beta - \alpha} \\
\sqrt{\beta - \alpha} & \sqrt{\beta + \alpha}
\end{bmatrix}
\]

is an orthogonal matrix which diagonalises \(\Omega\), and hence show how this idea may be used to find \(X = [\xi_x, \xi_y]^T\) with the correlation tensor \(\Omega\).

(c) Suppose that \((\varepsilon_1, \ldots, \varepsilon_n)\) is a vector of \(n\) uncorrelated Gaussian deviates drawn from the distribution \(N(0, 1)\). Use the previous idea to construct an \(n\)-dimensional random column vector \(X\) with correlation structure \(\Omega\) where \(\Omega\) is a positive definite \(n \times n\) array.

Answer

(a) Let \(X = [\xi_x, \xi_y]^T\) then \(\xi_x\) has variance \(\sigma_x^2\) and so we may write \(\xi_x = \sigma_x \varepsilon_1\) without any loss in generality. The task is now to find \(\xi_y\). The idea is to write \(\xi_y = \alpha \varepsilon_1 + \beta \varepsilon_2\). It now follows that

\[
E[\xi_x \xi_y] = E[\sigma_x \varepsilon_1 (\alpha \varepsilon_1 + \beta \varepsilon_2)] = \alpha \sigma_x
\]

\[
E[\xi_y \xi_y] = \alpha^2 + \beta^2
\]

Therefore, choose \(\alpha \sigma_x = \rho \sigma_x \sigma_y\) and \(\alpha^2 + \beta^2 = \sigma_y^2\). Thus \(\alpha = \rho \sigma_y\) and \(\beta^2 = \sigma_y^2(1 - \rho^2)\). One possible vector deviate is

\[
X = [\sigma_x \varepsilon_1, \rho \sigma_y \varepsilon_1 + \sqrt{1 - \rho^2} \sigma_y \varepsilon_2]^T.
\]
(b) To check that $Q$ is an orthogonal matrix, it is enough to observe that the two columns of $Q$ are orthogonal to each other, and that each column of $Q$ is a unit vector. Thus

$$Q^T \Omega Q = \frac{1}{2\beta} \begin{bmatrix} \sqrt{\beta + \alpha} & \sqrt{\beta - \alpha} \\ -\sqrt{\beta - \alpha} & \sqrt{\beta + \alpha} \end{bmatrix} \begin{bmatrix} \sigma_x^2 & \rho \sigma_x \sigma_y \\ \rho \sigma_x \sigma_y & \sigma_y^2 \end{bmatrix} \begin{bmatrix} \sqrt{\beta + \alpha} & -\sqrt{\beta - \alpha} \\ \sqrt{\beta - \alpha} & \sqrt{\beta + \alpha} \end{bmatrix}$$

Without substituting for $\alpha$ and $\beta$, the computation of $Q^T \Omega Q$ simplifies to

$$\frac{1}{2\beta} \begin{bmatrix} (\beta + \alpha)\sigma_x^2 + (\beta - \alpha)\sigma_y^2 + 2\rho \sigma_x \sigma_y \sqrt{\beta^2 - \alpha^2} & (\beta - \alpha)\sigma_x^2 - (\beta + \alpha)\sigma_y^2 + 2\rho \sigma_x \sigma_y \sqrt{\beta^2 - \alpha^2} \\ (\sigma_y^2 - \sigma_x^2) \sqrt{\beta^2 - \alpha^2} + 2\alpha \rho \sigma_x \sigma_y & (\beta + \alpha)\sigma_x^2 - (\beta - \alpha)\sigma_y^2 - 2\rho \sigma_x \sigma_y \sqrt{\beta^2 - \alpha^2} \end{bmatrix}.$$ 

We need to demonstrate that this is a diagonal matrix. Consider therefore

$$(\sigma_y^2 - \sigma_x^2) \sqrt{\beta^2 - \alpha^2} + 2\alpha \rho \sigma_x \sigma_y = -2\alpha \sqrt{\beta^2 - \alpha^2} + 2\alpha \rho \sigma_x \sigma_y = 2\alpha(\rho \sigma_x \sigma_y - \sqrt{\beta^2 - \alpha^2}).$$

It follows directly from the definition of $\beta$ that this entry is zero. Consequently, $Q^T \Omega Q$ is a diagonal matrix. By noting that, first that $\sigma_x^2 - \sigma_y^2 = 2\alpha$ and, second that $\rho \sigma_x \sigma_y = \sqrt{\beta^2 - \alpha^2}$, the array $Q^T \Omega Q$ now becomes

$$\frac{1}{2\beta} \begin{bmatrix} \beta(\sigma_x^2 + \sigma_y^2) + 2\alpha^2 + 2\rho \sigma_x \sigma_y \sqrt{\beta^2 - \alpha^2} & 0 \\ 0 & \beta(\sigma_y^2 + \sigma_x^2) - 2\alpha^2 - 2\rho \sigma_x \sigma_y \sqrt{\beta^2 - \alpha^2} \end{bmatrix}$$

$$= \frac{1}{2\beta} \begin{bmatrix} \beta(\sigma_x^2 + \sigma_y^2) + 2\beta^2 & 0 \\ 0 & \beta(\sigma_y^2 + \sigma_x^2) - 2\beta^2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \sigma_x^2 + \sigma_y^2 + 2\beta & 0 \\ 0 & \sigma_y^2 + \sigma_x^2 - 2\beta \end{bmatrix}$$

It is again obvious from the definition of $\beta$ that $\sigma_y^2 + \sigma_x^2 \geq 2\beta$ and so the entries of this diagonal matrix are non-negative. Let

$$Y = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\sigma_x^2 + \sigma_y^2 + 2\beta} \varepsilon_1, \sqrt{\sigma_x^2 + \sigma_y^2 - 2\beta} \varepsilon_2 \end{bmatrix}^T$$

where $\varepsilon_1 \sim N(0, 1)$ and $\varepsilon_2 \sim N(0, 1)$, then the random variable $X = QY$ has mean value zero and covariance tensor $Q^T \Omega Q$.

(c) Since $\Omega$ is a symmetric positive definite $n \times n$ array, it is possible to find an $n \times n$ orthogonal matrix $Q$ such that $\Omega = QDQ^T$ in which $D$ is a diagonal matrix whose entries are the eigenvalues of $\Omega$ in some order. Since $\Omega$ is positive definite, then each entry of $D$ is positive. Let $Y = [\sqrt{\lambda_1} \varepsilon_1, \cdots, \sqrt{\lambda_n} \varepsilon_n]^T$ where $\lambda_k > 0$ is the $k$-th diagonal entry of $D$. Consider the properties of $X = QY$. Clearly

$$E[X] = E[QY] = Q E[Y] = 0, $$

$$E[X X^T] = E[Q YY^T Q^T] = Q E[Y Y^T] Q^T = Q D Q^T = \Omega.$$
Thus \( X \) has the required properties.

Q 4. Let \( X \) be normally distributed with mean zero and unit standard deviation, then \( Y = X^2 \) is said to be \( \chi^2 \) distributed with one degree of freedom.

(a) Show that \( Y \) is Gamma distribution with \( \lambda = \rho = 1/2 \).

(b) What is now the distribution of \( Z = aX^2 \) if \( a > 0 \) and \( X \sim N(0, \sigma^2) \).

(b) If \( X_1, \cdots, X_n \) are \( n \) independent Gaussian distributed random variables with mean zero and unit standard deviation, what is the distribution of \( Y = X^T X \) where \( X \) is the \( n \) dimensional column vector with \( k \)-th entry \( X_k \).

Answer

(a) Since \( Y = X^2 \) then clearly \( Y \geq 0 \) and so every non-zero value of \( Y \) arises from either \( X \) or \( -X \). Therefore, the density of \( Y \) is

\[
f_Y(y) = 2f_X(x) \frac{dX}{dY} = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{2} y^{-1/2} = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}.
\]

Evidently the distribution function for \( Y \) is the special case of the Gamma distribution in which \( \lambda = \rho = 1/2 \).

(b) Since \( Z = aX^2 \) with \( a > 0 \) then clearly \( Z \geq 0 \) and so once again every non-zero value of \( Z \) arises from either \( X \) or \( -X \). The density of \( Z \) is now

\[
f_Z(z) = 2f_X(x) \frac{dX}{dZ} = \frac{2}{\sqrt{2\pi} \sqrt{a}} e^{-z^2/(2a)} \frac{1}{2\sqrt{a}} z^{-1/2} = \frac{1}{\sqrt{2\pi} \sqrt{a}} z^{-1/2} e^{-z^2/(2a\sigma^2)}.
\]

The distribution function for \( Z \) is now the Gamma distribution with \( \rho = 1/2 \) and \( \lambda = (2a\sigma^2)^{-1} \).

(c) If \( X_1, \cdots, X_n \) are each normally distributed with mean zero and unit standard deviation, then \( Y = X_1^2 + X_2^2 + \cdots + X_n^2 \) is likewise Gamma distributed with parameters \( \lambda = 1/2 \) and \( \rho = n/2 \). This result follows from the previous example. In this case we say that \( Y \) is \( \chi^2 \) distributed with \( n \) degrees of freedom.

The chi-squared distribution with \( n \) degrees of freedom plays an important role in statistical hypothesis testing in which deviates are assigned to “bins”.

Q 5. Calculate the bounded variation (when it exists) for the functions

(a) \( f(x) = |x| \quad x \in [-1, 2] \)  
(b) \( g(x) = \log x \quad x \in (0, 1] \)  
(c) \( h(x) = 2x^3 + 3x^2 - 12x + 5 \quad x \in [-3, 2] \)  
(d) \( k(x) = 2 \sin 2x \quad x \in [\pi/12, 5\pi/4] \)  
(e) \( n(x) = H(x) - H(x - 1) \quad x \in \mathbb{R} \)  
(f) \( m(x) = (\sin 3x)/x \quad x \in [-\pi/3, \pi/3] \).
Answer

The simplest approach here is to draw each function.

(a) The bounded variation of \( f(x) = |x| \) for \( x \in [-1, 2] \) is 1 + 2 = 3.

(b) The function \( g(x) = \log x \to -\infty \) as \( x \to 0^+ \).

(c) Here we need to find the stationary values of \( h(x) = 2x^3 + 3x^2 - 12x + 5 \) for \( x \in [-3, 2] \). Clearly \( \frac{dh}{dx} = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x + 2)(x - 1) \) and so \( h(x) \) has turning values at \( x = -2 \) and \( x = 1 \). Thus \( h(-3) = 4, h(-2) = 25, h(1) = -2 \) and \( h(2) = 9 \) and so the bounded variation is

\[
|4 - 25| + |25 - (-2)| + |(-2) - 9| = 59.
\]

(d) We need to draw the function in our mind. The points of interest are \( h(\pi/12) = 2 \sin(\pi/6) = 1, h(\pi/4) = 2 \sin(\pi/2) = 2, h(3\pi/4) = 2 \sin(3\pi/2) = -2 \) and \( h(5\pi/4) = 2 \sin(5\pi/2) = 2 \). Thus the variation is 1 + 4 + 4 = 9.

(e) The Heaviside function or step function \( H(x) \) jumps from zero to 1 as \( x \) passes through \( x = 0 \) while the step function \( H(x - 1) \) jumps from zero to 1 as \( x \) passes through \( x = 1 \). Thus \( n(x) \) is the top hat function and its variation is 2.

(f) The function \( m(x) = (\sin 3x)/x \) is zero at the end points of the interval \([-\pi/3, \pi/3]\). It takes the value 3 as \( x \to 0 \) and therefore the variation is 6.

Q 6. Use the definition of the Riemann integral to demonstrate that

\[
(a) \int_0^1 x \, dx = \frac{1}{2}, \quad (b) \int_0^1 x^2 \, dx = \frac{1}{3}.
\]

You will find useful the formulae \( \sum_{k=1}^n k = n(n + 1)/2 \) and \( \sum_{k=1}^n k^2 = n(n + 1)(2n + 1)/6 \).

**Answer**

Choose a uniform dissection of \([0, 1]\) in which the interval is divided into \( n \) sub-intervals of length \( 1/n \) by the points \( x_k = k/n \) where \( 0 \leq k \leq n \).

(a) In the first example we shall pedantic and take a different value of the function in each subinterval of the dissection in order to illustrate in this simple case that the value of the integral is independent of the choice of dissection point. In the following partial sum \( \lambda_k \in [0, 1] \) so that the partial sum is

\[
S_n = \frac{1}{n} \sum_{k=1}^n f(\xi_k) = \frac{1}{n} \sum_{k=1}^n x_{k-1} \lambda_k + (1 - \lambda_k) x_k = \frac{1}{n} \sum_{k=1}^n \frac{(k - 1)}{n} \lambda_k + \frac{k}{n} (1 - \lambda_k).
\]
By simple algebra $S_n$ becomes

$$S_n = -\frac{1}{n^2} \sum_{k=1}^{n} \lambda_k + \frac{1}{n^2} \frac{n(n+1)}{2} = -\frac{1}{n^2} \sum_{k=1}^{n} \lambda_k + \frac{1}{2} \left(1 + \frac{1}{n}\right).$$

Clearly $0 \leq \sum_{k=1}^{n} \lambda_k \leq n$ and therefore

$$\frac{1}{2} \left(1 - \frac{1}{n}\right) \leq S_n \leq \frac{1}{2} \left(1 + \frac{3}{n}\right) \quad \Rightarrow \quad \lim_{n \to \infty} S_n = \frac{1}{2}.$$

(b) Here we take the left hand endpoint of each interval to deduce that

$$\int_0^1 x^2 \, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{n^2} = \lim_{n \to \infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \lim_{n \to \infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}.$$

Q 7. The functions $f$, $g$ and $h$ are defined on $[0,1]$ by the formulae

(a) $f(x) = \begin{cases} x \sin(\pi/x) & x > 0 \\ 0 & x = 0 \end{cases}$

(b) $g(x) = \begin{cases} x^2 \sin(\pi/x) & x > 0 \\ 0 & x = 0 \end{cases}$

(c) $h(x) = \begin{cases} (x/\log x) \sin(\pi/x) & x > 0 \\ 0 & x = 0,1 \end{cases}$

Determine which functions have bounded variation on the interval $[0,1]$.

**Answer**

Either provide a proof that $f$ is a function of bounded variation over $[0,1]$ or provide a dissection $D_n$ over which it is clear that $f$ is not a function of bounded variation as $n \to \infty$.

(a) Let $D_n$ be the dissection of $[0,1]$ with nodes $x_0 = 0$, $x_k = 2/(2n - 2k + 1)$ when $1 \leq k \leq n - 1$ and $x_n = 1$. Clearly $f(x_0) = f(x_n) = 0$ while

$$f(x_k) = \frac{2}{2n - 2k + 1} \sin((n-k)\pi + \pi/2) = \frac{2(-1)^{n-k}}{2n - 2k + 1}.$$  

The variation of $f$ over $D_n$ is therefore

$$V(f) = \sum_{k=0}^{n} \frac{2(-1)^{n-k}}{2n - 2k + 1} = \sum_{k=0}^{n} \frac{2}{2n - 2k + 1} = \sum_{k=0}^{n} \frac{2}{2k + 1}.$$  

This series diverges, and so $f$ is not of bounded variation over $[0,1]$. 

(b) The stationary values of $g$ occur where
\[
\frac{dg}{dx} = 2x \sin(\pi/x) - \pi \cos(\pi/x) = 0 \rightarrow \tan(\pi/x) = \pi/(2x).
\]
The solution to this equation is $x_1 = \pi/\theta_1, x_2 = \pi/\theta_2, \ldots$ where $\tan(\theta_k) = \theta_k/2$. Let $D$ be the dissection based on the points $1 = x_0 > x_1 > x_2 > \cdots > 0$, then $g$ either decreases or increases between the nodes of $D$. Consequently, the variation of $g$ over $D$ is
\[
V(g) = \sum_{k=1}^{\infty} \frac{\pi^2}{\theta_k} \sin \theta_k \leq \sum_{k=1}^{\infty} \frac{\pi^2}{\theta_k^2}.
\]
It is easy to demonstrate that $\theta_1 > \pi/4$, $\theta_2 > 5\pi/4$ and in general that $\theta_k > k\pi - 3\pi/4$. Therefore $\pi/\theta_k < 4/(4k - 3)$. Finally, it follows from (1) that
\[
V(g) \leq \sum_{k=1}^{\infty} \frac{\pi^2}{\theta_k^2} \leq \sum_{k=1}^{\infty} \frac{16}{(4k - 3)^2} < \infty.
\]
This series converges, and so $g$ is of bounded variation over $[0, 1]$.

(c) Let $D_n$ be the dissection of $[0, 1]$ with nodes $x_0 = 0, x_k = 2/(2n - 2k + 1)$ when $1 \leq k \leq n - 1$ and $x_n = 1$. Clearly $h(x_0) = h(x_n) = 0$ while
\[
h(x_k) = \frac{2}{(2n - 2k + 1) \log(2/(2n - 2k + 1))} \sin((n - k)\pi + \pi/2)
= \frac{2(-1)^{n-k}}{(2n - 2k + 1) \log(2/(2n - 2k + 1))}
\]
The variation of $h$ over $D_n$ therefore satisfies
\[
V(h) \geq \sum_{k=1}^{n-1} \left| \frac{2(-1)^{n-k}}{(2n - 2k + 1) \log(2/(2n - 2k + 1))} \right|
= \sum_{k=1}^{n-1} \frac{2}{(2n - 2k + 1) \log(n - k + 1/2)}
= \sum_{k=1}^{n-1} \frac{2}{(2k + 1) \log(k + 1/2)}
\]
This series can be condensed. The condensed series will diverge, and so $h$ is not a function of bounded variation over $[0, 1]$.

Q 8. Prove that
\[
\int_{a}^{b} (dW_t)^2 G(t) = \int_{a}^{b} G(t) \, dt
\]
Answer

Let $D_n$ denote the dissection $a = t_0 < t_1 < \cdots < t_n = b$ of the finite interval $[a, b]$ and let

$$
\Phi = \int_a^b (dW_t)^2 G(t) = \int_a^b G(t) \, dt.
$$

The value of $\Phi$ is based on the mean square limiting value of the partial sum

$$
S_n = \sum_{k=1}^{n} G(t_{k-1}) (W_k - W_{k-1})^2
$$

Since $(W_k - W_{k-1}) = \sqrt{t_k - t_{k-1}} \varepsilon_k$ where $\varepsilon_k \sim N(0, 1)$ then

$$
\lim_{n \to \infty} E[S_n] = \lim_{n \to \infty} E \left[ \sum_{k=1}^{n} G(t_{k-1}) (t_k - t_{k-1}) \varepsilon_k^2 \right]
$$

$$
= \lim_{n \to \infty} \sum_{k=1}^{n} G(t_{k-1}) (t_k - t_{k-1}) E[\varepsilon_k^2]
$$

$$
= \lim_{n \to \infty} \sum_{k=1}^{n} G(t_{k-1}) (t_k - t_{k-1})
$$

$$
= \int_a^b G(t) \, dt.
$$

To demonstrate mean square convergence one needs to show that

$$
\lim_{n \to \infty} E \left[ \left( \sum_{k=1}^{n} G(t_{k-1}) (t_k - t_{k-1}) (\varepsilon_k^2 - 1) \right)^2 \right] = 0.
$$

The calculation proceeds as follows.

$$
\lim_{n \to \infty} E \left[ \left( \sum_{k=1}^{n} G(t_{k-1}) (t_k - t_{k-1}) (\varepsilon_k^2 - 1) \right)^2 \right]
$$

$$
= \lim_{n \to \infty} E \left[ \sum_{k=1}^{n} \sum_{j=1}^{n} G(t_{k-1}) (t_k - t_{k-1}) (\varepsilon_k^2 - 1) G(t_{j-1}) (t_j - t_{j-1}) (\varepsilon_j^2 - 1) \right]
$$

$$
= \lim_{n \to \infty} \sum_{k=1}^{n} \sum_{j=1}^{n} G(t_{k-1}) G(t_{j-1}) (t_k - t_{k-1}) (t_j - t_{j-1}) E[ (\varepsilon_k^2 - 1) (\varepsilon_j^2 - 1) ].
$$

The value of this expected value is zero whenever $j \neq k$. Thus expression (2) simplifies to

$$
\lim_{n \to \infty} E \left[ \left( \sum_{k=1}^{n} G(t_{k-1}) (t_k - t_{k-1}) (\varepsilon_k^2 - 1) \right)^2 \right]
$$

$$
= \lim_{n \to \infty} \sum_{k=1}^{n} G^2(t_{k-1}) (t_k - t_{k-1})^2 E[\varepsilon_k^4 - 2\varepsilon_k^2 + 1]
$$

$$
= 2 \lim_{n \to \infty} \sum_{k=1}^{n} G^2(t_{k-1}) (t_k - t_{k-1})^2 = 0.
$$
Q 9. Prove that

\[ \int_{a}^{b} W^n dW_t = \frac{1}{n+1} \left[ W(b)^{n+1} - W(a)^{n+1} \right] - \frac{n}{2} \int_{a}^{b} W^{n-1} \, dt. \]

**Answer**

First apply the identity

\[ \int_{a}^{b} f(t, W_t) \, dW_t = \int_{a}^{b} f(t, W_t) \, dW_t + \frac{1}{2} \int_{a}^{b} \frac{\partial f(t, W_t)}{\partial W_t} \, dt. \]

with \( f(t, W_t) = W^n_t \) to obtain

\[ \int_{a}^{b} W^n_t \, dW_t = \int_{a}^{b} W^n_t \, dW_t + \frac{n}{2} \int_{a}^{b} W^{n-1} \, dt. \]

Now apply the identity

\[ \int_{a}^{b} \frac{\partial g(t, W_t)}{\partial W_t} \, dW_t = g(b, W(b)) - g(a, W(a)) - \int_{a}^{b} \frac{\partial g}{\partial t} \, dt. \]

with \( g(t, W_t) = W^n_t \) to obtain

\[ \int_{a}^{b} W^n_t \, dW_t = \frac{1}{n+1} \left[ W^{n+1}(b) - W^{n+1}(a) \right]. \]

The final result is therefore

\[ \int_{a}^{b} W^n_t \, dW_t = \frac{1}{n+1} \left[ W^{n+1}(b) - W^{n+1}(a) \right] - \frac{n}{2} \int_{a}^{b} W^{n-1} \, dt. \]

Q 10. The connection between the Stratonovich and Ito integrals relied on the claim that

\[ \lim_{n \to \infty} \mathbb{E} \left[ \sum_{k=1}^{n} \frac{\partial f(t_{k-1}, W_{k-1})}{\partial W} \left( (W_{k-1/2} - W_{k-1})(W_k - W_{k-1}) - \frac{t_k - t_{k-1}}{2} \right) \right]^2 = 0 \]

provided \( \mathbb{E}[|\partial f(t, W)|^2] \) is integrable over the interval \([a, b]\). Verify this unsubstantiated claim.

**Answer**

The calculation begins with the consideration of

\[ \psi_n = \left[ \sum_{k=1}^{n} \frac{\partial f(t_{k-1}, W_{k-1})}{\partial W} \left( (W_{k-1/2} - W_{k-1})(W_k - W_{k-1}) - \frac{t_k - t_{k-1}}{2} \right) \right]^2. \]

This function is expanded out to give the double summation

\[ \psi = \sum_{j,k=1}^{n} \left[ \frac{\partial f(t_{j-1}, W_{j-1})}{\partial W} \left( (W_{j-1/2} - W_{j-1})(W_j - W_{j-1}) - \frac{t_j - t_{j-1}}{2} \right) \right] \left[ \frac{\partial f(t_{k-1}, W_{k-1})}{\partial W} \left( (W_{k-1/2} - W_{k-1})(W_k - W_{k-1}) - \frac{t_k - t_{k-1}}{2} \right) \right]. \]
which is now divided into the case $j = k$ and the case $j \neq k$ to get
\[
\psi = \sum_{k=1}^{n} \frac{\partial f(t_{k-1}W_{k-1})}{\partial W} \left[ (W_{k-1/2} - W_{k-1})(W_{k} - W_{k-1}) - \frac{t_k - t_{k-1}}{2} \right]^2 \\
+ 2 \sum_{1 \leq j < k \leq n} \frac{\partial f(t_{j-1}W_{j-1})}{\partial W} \left( (W_{j-1/2} - W_{j-1})(W_{j} - W_{j-1}) - \frac{t_j - t_{j-1}}{2} \right) \frac{\partial f(t_{k-1}W_{k-1})}{\partial W} \left( (W_{k-1/2} - W_{k-1})(W_{k} - W_{k-1}) - \frac{t_k - t_{k-1}}{2} \right)
\]

When $j \neq k$ then $(W_{j-1/2} - W_{j-1})(W_{j} - W_{j-1}) - (t_j - t_{j-1})/2$ and $(W_{k-1/2} - W_{k-1})(W_{k} - W_{k-1}) - (t_k - t_{k-1})/2$ are independent random deviates with mean value zero. Therefore the expectation of the double sum is zero. To appreciate why this is the case consider the expectation being taken term-by-term starting at $k = n$ and working downwards. Define $\varepsilon_{k-1} = W_{k-1/2} - W_{k-1}$ then
\[
\left[ (W_{k-1/2} - W_{k-1})(W_{k} - W_{k-1}) - \frac{t_k - t_{k-1}}{2} \right]^2 \\
= \left[ -\varepsilon_{k-1}^2 + \varepsilon_{k-1/2}^2 + \frac{t_k - t_{k-1}}{2} \right]^2 \\
= \varepsilon_{k-1}^2 - \varepsilon_{k-1}^2 + \varepsilon_{k-1/2}^2 + \frac{t_k - t_{k-1}}{2} - \left( W_{k-1} - W_{k-1}\right) \left( -\varepsilon_{k-1}^2 + \varepsilon_{k-1/2}^2 \right) - 2\varepsilon_{k-1}^3 \varepsilon_{k-1}
\]

\[
\mathbb{E}[W_{k-1/2} - W_{k-1})(W_{k} - W_{k-1}) - (t_k - t_{k-1})/2]^2 = 3(t_k - t_{k-1})^2/4 
\]

and therefore
\[
\mathbb{E}[\psi_n] \leq \frac{3}{4} \max |t_k - t_{k-1}| \sum_{k=1}^{n} \left[ \frac{\partial f(t_{k-1}W_{k-1})}{\partial W} \right]^2 (t_k - t_{k-1}) \to 0 
\]
as $n \to \infty$.

Q 11. Compute the value of
\[
\Phi = \int_{a}^{b} W_t \, dW_t
\]
where the definition of the integral is based on the choice $\xi_k = (1 - \lambda)t_{k-1} + \lambda t_k$ and $\lambda \in [0, 1]$.

**Answer**

Let $D_n$ denote the dissection $a = t_0 < t_1 < \cdots < t_n = b$ of the finite interval $[a, b]$ and let
\[
\Phi = \int_{a}^{b} W_t \, dW_t
\]
where the limiting procedure is based on the choice $\xi_k = (1 - \lambda)t_{k-1} + \lambda t_k$ with $\lambda \in [0, 1]$. The task is to find $\Phi$ such that
\[
\lim_{n \to \infty} \mathbb{E} \left[ \left( \Phi - \sum_{k=1}^{n} W(\xi_k)(W_{k} - W_{k-1}) \right)^2 \right] 
\]
(4)

The algebraic manipulations
\[
2W_{k+\lambda-1}W_k = W_{k+\lambda-1}^2 + W_k^2 - (W_{k+\lambda-1} - W_k)^2 \\
2W_{k+\lambda-1}W_{k-1} = W_{k+\lambda-1}^2 + W_{k-1}^2 - (W_{k+\lambda-1} - W_{k-1})^2 
\]
are required to proceed further. Start by subtracting the first expression from the second, and then sum from \( k = 1 \) to \( k = n \) to obtain

\[
2 \sum_{k=1}^{n} W_{k+\lambda-1} (W_k - W_{k-1}) = \sum_{k=1}^{n} W_k^2 - (W_{k+\lambda-1} - W_k)^2 - W_{k-1}^2 + (W_{k+\lambda-1} - W_{k-1})^2
\]

\[
= W^2(b) - W^2(a) + \sum_{k=1}^{n} (W_{k+\lambda-1} - W_k)^2 - (W_{k+\lambda-1} - W_k)^2
\]

\[
= W^2(b) - W^2(a) + \sum_{k=1}^{n} (\xi_k - t_{k-1}) \varepsilon_k^2 - (t_k - \xi_k) \eta_k^2
\]

where \( \varepsilon_k \) and \( \eta_k \) are uncorrelated \( N(0, 1) \) deviates. The expected value of the summation on the right hand side of this expression is \((2\lambda - 1)(b - a)\) and therefore the contention is that

\[
\Phi = \frac{1}{2} \left[ W^2(b) - W^2(a) + (2\lambda - 1)(b - a) \right].
\]

To prove this assertion, we substitute this expression for \( \Phi \) in equation (4). We need to show that

\[
\frac{1}{2} \lim_{n \to \infty} E \left[ \left( (2\lambda - 1)(b - a) - \lambda \sum_{k=1}^{n} \varepsilon_k^2(t_k - t_{k-1}) + (1 - \lambda) \sum_{k=1}^{n} \eta_k^2(t_k - t_{k-1}) \right)^2 \right] = 0
\]

The calculation proceeds as follows:

\[
\lim_{n \to \infty} E \left[ \left( (2\lambda - 1)(b - a) - \lambda \sum_{k=1}^{n} \varepsilon_k^2(t_k - t_{k-1}) + (1 - \lambda) \sum_{k=1}^{n} \eta_k^2(t_k - t_{k-1}) \right)^2 \right]
\]

\[
= \lim_{n \to \infty} E \left[ \left( \lambda \sum_{k=1}^{n} (\varepsilon_k^2 - 1)(t_k - t_{k-1}) - (1 - \lambda) \sum_{k=1}^{n} (\eta_k^2 - 1)(t_k - t_{k-1}) \right)^2 \right]
\]

\[
= \lim_{n \to \infty} E \left[ \lambda^2 \sum_{k=1}^{n} \sum_{j=1}^{n} (\varepsilon_k^2 - 1)(\varepsilon_j^2 - 1)(t_k - t_{k-1})(t_j - t_{j-1})
\]

\[
- 2\lambda(1 - \lambda) \sum_{k=1}^{n} \sum_{j=1}^{n} (\eta_k^2 - 1)(\eta_j^2 - 1)(t_k - t_{k-1})(t_j - t_{j-1})
\]

\[
+ (1 - \lambda)^2 \sum_{k=1}^{n} \sum_{j=1}^{n} (\eta_k^2 - 1)(\eta_j^2 - 1)(t_k - t_{k-1})(t_j - t_{j-1}) \right]
\]

In the usual way, if \( k \neq j \) then \( (\xi_j^2 - 1) \) and \( (\xi_k^2 - 1) \) are uncorrelated, otherwise they are correlated. Thus the previous argument continues with

\[
\lim_{n \to \infty} E \left[ \left( (2\lambda - 1)(b - a) - \lambda \sum_{k=1}^{n} \varepsilon_k^2(t_k - t_{k-1}) + (1 - \lambda) \sum_{k=1}^{n} \eta_k^2(t_k - t_{k-1}) \right)^2 \right]
\]

\[
= \lim_{n \to \infty} E \left[ \lambda^2 \sum_{k=1}^{n} (\varepsilon_k^2 - 1)^2(t_k - t_{k-1})^2 + (1 - \lambda)^2 \sum_{k=1}^{n} (\eta_k^2 - 1)^2(t_k - t_{k-1})^2 \right]
\]

As has been seen previously, \( E[(\varepsilon_k^2 - 1)^2] = E[\varepsilon_k^2 - 2\varepsilon_k + 1] = 2 \) with an identical argument for
\[ E[(\eta_k^2 - 1)^2]. \] Consequently,

\[
\lim_{n \to \infty} E \left[ \left( (2\lambda - 1)(b-a) - \lambda \sum_{k=1}^n \varepsilon_k^2(t_k - t_{k-1}) + (1 - \lambda) \sum_{k=1}^n \eta_k^2(t_k - t_{k-1}) \right)^2 \right] \\
= 2[\lambda^2 + (1 - \lambda)^2] \lim_{n \to \infty} \sum_{k=1}^n (t_k - t_{k-1})^2 = 0.
\]

Hence it has been proved that

\[
\int_a^b W_t \, dW_t = \frac{1}{2} \left[ W^2(b) - W^2(a) + (2\lambda - 1)(b-a) \right]
\]

with the choice \( \xi_k = (1 - \lambda)t_{k-1} + \lambda t_k \) and \( \lambda \in [0,1] \).

**Q 12.** Solve the stochastic differential equation

\[ dx_t = a(t) \, dt + b(t) \, dW_t, \quad x_0 = X_0 \]

where \( X_0 \) is constant. Show that the solution \( x_t \) is a Gaussian deviate and find its mean and variance.

**Answer**

The formal solution to this problem is

\[ x(t) = X_0 + \int_0^t a(s) \, ds + \int_0^t b(s) \, dW_s \]

where the second integral is to be interpreted as an Ito integral. The fact that \( x(t) \) is a Gaussian deviate stems from the fact that the stochastic part of \( x \) is simply a weighted combination of Gaussian deviates, and is therefore a Gaussian deviate. The mean is

\[ \mu(t) = X_0 + \int_0^t a(s) \, ds \]

and the variance is

\[ E[ (x - \mu)^2] = E \left[ \int_0^t b(s) \, dW_s \int_0^t b(s) \, dW_s \right] = \int_0^t b^2(s) \, ds. \]

**Q 13.** The stochastic integrals \( I_{01} \) and \( I_{10} \) defined by

\[ I_{10} = \int_a^b \int_0^s dW_u \, ds, \quad I_{01} = \int_a^b \int_0^s du \, dW_s. \]

occur frequently in the numerical solution of stochastic differential equations. Show that

\[ I_{10} + I_{01} = (b - a)(W_b - W_a). \]

**Answer**
The double integrals for \( I_{10} \) and \( I_{01} \) may be converted to the single integrals

\[
I_{10} = \int_a^b \int_a^s dW_u \, ds = \int_a^b (W_s - W_a) \, ds,
\]
\[
I_{01} = \int_a^b \int_a^s du \, dW_s = \int_a^b (s - a) \, dW_s.
\]

It is clear that \( I_{10} \) is a Riemann integral and \( I_{01} \) is a Riemann-Stieltjes integral. Both integrals may be computed using the standard rules of integral calculus. Thus

\[
I_{01} = \int_a^b (s - a) \, dW_s
\]
\[
= \left[ (s - a)(W_s - W_a) \right]_a^b - \int_a^b (W_s - W_a) \, ds
\]
\[
= (b - a)(W_b - W_a) - I_{10}.
\]

Thus \( I_{01} + I_{10} = (b - a)(W_b - W_a) \).

---

**Q 14.** The stochastic integrals \( I_{111} \) defined by

\[
I_{111} = \int_a^t \int_a^s \int_a^u dW_w \, dW_u \, dW_s
\]

occurs frequently in the numerical solution of stochastic differential equations. Show that

\[
I_{111} = \frac{(W_b - W_a)}{6} \left[ (W_b - W_a)^2 - 3(b - a) \right].
\]

**Answer**

The evaluation of \( I_{111} \) proceeds in a number of steps, the first of which is the conversion of the triple integral to the double integral

\[
I_{111} = \int_a^b \int_a^s \int_a^u dW_w \, dW_u \, dW_s = \int_a^b \int_a^s (W_u - W_a) \, dW_u \, dW_s.
\]

where \( W_s = W(s) \) etc. The double integral may now be further integrated to yield

\[
I_{111} = \frac{1}{2} \int_a^b \left[ (W_s - W_a)^2 - (s - a) \right] \, dW_s
\]
\[
= \left[ \frac{1}{6} (W_s - W_a)^3 \right]_a^b - \frac{1}{2} \int_a^b (W_s - W_a) \, ds - \frac{1}{2} \int_a^b (s - a) \, dW_s
\]
\[
= \frac{1}{6} (W_b - W_a)^3 - \frac{1}{2} \left( I_{10} + I_{01} \right)
\]

where \( I_{10} \) and \( I_{01} \) are defined in the previous problem which asserts that

\[
I_{10} + I_{01} = (b - a)(W_b - W_a).
\]
The final conclusion is therefore that

\[ I_{111} = \frac{(W_b - W_a)}{6} \left[ (W_b - W_a)^2 - 3(b-a) \right]. \]

Q 15. Show that the value of the Riemann integral

\[ I_{10} = \int_a^b \int_0^s dW_a ds \]

may be simulated as a Gaussian deviate with mean value zero, variance \((b-a)^3/3\) and such that its correlation with \((W_b - W_a)\) is \((b-a)^2/2\).

Answer

It has already been shown that

\[ \xi = I_{10} = \int_a^b \int_a^s dW_a ds = \int_a^b (W_s - W_a) ds. \]

First, it is clear that \(\xi\) is a Gaussian process since it is simply the sum of Gaussian processes. The mean value of \(\xi\) is zero and its variance is

\[
\begin{align*}
E[\xi^2] &= E \left[ \int_a^b (W_s - W_a) ds \int_a^b (W_u - W_a) du \right] \\
&= E \left[ \int_a^b \int_a^b (W_s - W_a) (W_u - W_a) ds du \right] \\
&= \int_a^b \int_a^b E \left[ (W_s - W_a) (W_u - W_a) \right] ds du \\
&= \int_a^b \int_a^b \min(s-a,u-a) ds du \\
&= 2 \int_a^b \int_a^u (s-a) ds du \\
&= \int_a^b (u-a)^2 du = \frac{(b-a)^3}{3}.
\end{align*}
\]

The correlation between \((W_b - W_a)\) and \(\xi\) is given by

\[
E \left[ (W_b - W_a) \int_a^b (W_s - W_a) ds \right] = \int_a^b E \left[ (W_b - W_a) (W_s - W_a) \right] ds
\]

\[ = \int_a^b (s-a) ds = \frac{(b-a)^2}{2}. \]

In conclusion, \(\xi = I_{10}\) is simulated as a Gaussian deviate with mean value zero, variance \((b-a)^3/3\) and such that it has correlation \((b-a)^2/2\) with \((W_b - W_a)\).

Q 16. Solve the degenerate Ornstein-Uhlenbeck stochastic initial value problem

\[ dx = -\alpha x + \sigma dW, \quad x(0) = x_0 \]
in which $\alpha$ and $\sigma$ are positive constants and $x_0$ is a random variable. Deduce that

$$E[X] = E[x_0] e^{-\alpha t}, \quad V[X] = V[x_0] e^{-2\alpha t} + \frac{\sigma^2}{2\alpha} [1 - e^{-2\alpha t}].$$

**Answer**

The solution may be obtained by applying Ito’s lemma to the function $F = xe^{\alpha t}$. Ito’s lemma for this SDE is

$$dF = \left[ F_t(t,x) - \alpha x F_x(t,x) \right] dt + \sigma F_x(t,x) dW_t$$

and in this case yields

$$dF = \sigma e^{\alpha t} dW_t.$$ 

Thus the solution is

$$x(t) = x_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(s-t)} dW_s.$$ 

The mean solution is

$$E[x(t)] = E[x_0] e^{-\alpha t}.$$ 

Consequently,

$$x(t) - E[x(t)] = (x_0 - E[x_0]) e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(s-t)} dW_s.$$ 

Assuming that $x_0$ is uncorrelated with the Wiener process, it follows that

$$V[x(t)] = V[x_0] e^{-2\alpha t} + \sigma^2 \left[ \int_0^t e^{-\alpha(s-t)} dW_s \int_0^t e^{-\alpha(u-t)} dW_u \right]$$

$$= V[x_0] e^{-2\alpha t} + \sigma^2 \int_0^t \int_0^t e^{-\alpha(s-t)} e^{-\alpha(u-t)} E[dW_s dW_u]$$

$$= V[x_0] e^{-2\alpha t} + \sigma^2 \int_0^t e^{-2\alpha(s-t)} ds$$

$$= V[x_0] e^{-2\alpha t} + \frac{\sigma^2}{2\alpha} [1 - e^{-2\alpha t}].$$

---

**Q 17.** It is given that the instantaneous rate of interest satisfies the equation

$$dr = \mu(t,r) dt + \sqrt{g(t,r)} dW.$$ \hspace{1cm} (5)

Let $B(t,R)$ be the value of a zero-coupon bond at time $t$ paying one dollar at maturity $T$. Show that $B(t,R)$ is the solution of the stochastic differential equation

$$\frac{\partial B}{\partial t} + \mu(t,r) \frac{\partial B}{\partial r} + \frac{g(t,r)}{2} \frac{\partial^2 B}{\partial r^2} = rB$$ \hspace{1cm} (6)

with terminal condition $B(T,r) = 1$. 


(a) The CIR equation takes the form of equation (8) with \( \mu(t, r) = \alpha(\theta - r) \) and \( g(t, r) = \sigma^2 r \). It is given that the anzatz \( B(t, r) = \exp[\beta_0(T - r) + \beta_1(T - t)r] \) satisfies equation (6) provided the coefficient functions \( \beta_0(T - t) \) and \( \beta_1(T - t) \) satisfy a pair of ordinary differential equations. Determine these equations with their boundary conditions.

(b) Solve these ordinary differential equations and deduce the value of \( B(t, R) \) where \( R \) is the spot rate of interest.

**Answer**

Ito’s lemma for \( B(t, r) \) gives

\[
\frac{dB}{dt} = \frac{\partial B}{\partial t} dt + \frac{\partial B}{\partial r} dr + \frac{g(t, r)}{2} \frac{\partial^2 B}{\partial r^2} dt \quad \rightarrow \quad E[dB] = \left( \frac{\partial B}{\partial t} + \mu(t, r) \frac{\partial B}{\partial r} + \frac{g(t, r)}{2} \frac{\partial^2 B}{\partial r^2} \right) dt
\]

But \( E[dB] = rB dt \) and therefore \( B(t, r) \) satisfies the partial differential equation

\[
\frac{\partial B}{\partial t} + \mu(t, r) \frac{\partial B}{\partial r} + \frac{g(t, r)}{2} \frac{\partial^2 B}{\partial r^2} = rB.
\]

**Q 18.** It is given that the solution of the initial value problem

\[
dx = -\alpha x + \sigma x \, dW, \quad x(0) = x_0
\]

in which \( \alpha \) and \( \sigma \) are positive constants is

\[
x(t) = x(0) \exp\left[ -(\alpha + \sigma^2/2)t + \sigma W(t) \right]
\]

Show that

\[
E[X] = x_0 e^{-\alpha t}, \quad V[X] = e^{-2\alpha t}(e^{\sigma^2 t} - 1) x_0^2.
\]

**Answer**

The expected value of \( X \) is

\[
E[X] = E\left[x(0) \exp\left[ -(\alpha + \sigma^2/2)t + \sigma W(t) \right]\right] = x(0) e^{-(\alpha + \sigma^2/2)t} E[e^{-\sigma W(t)}]
\]

The task is now to compute

\[
E[e^{-\sigma W(t)}] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\sigma W} e^{-W^2/2t} dW
\]

\[
= \frac{1}{\sqrt{2\pi t}} e^{\sigma^2 t/2} \int_{-\infty}^{\infty} e^{-(W+\sigma t)^2/2t} dW = e^{\sigma^2 t/2}.
\]

Consequently it follows that

\[
E[X] = x(0) e^{-(\alpha + \sigma^2/2)t} e^{\sigma^2 t/2} = x(0) e^{-\alpha t}.
\]
The variance of $X$ is

$$V[X] = E \left[ \left( x(0) \exp[-(\alpha + \sigma^2/2)t + \sigma W(t)] - x(0) e^{-(\alpha t)} \right)^2 \right]$$

$$= x^2(0) e^{-2\alpha t} E \left[ \left( \exp[-\sigma^2 t/2 + \sigma W(t)] - 1 \right)^2 \right]$$

$$= x^2(0) e^{-2\alpha t - \sigma^2 t} E \left[ \left( e^{\sigma W(t)} - e^{\sigma^2 t/2} \right)^2 \right]$$

$$= x^2(0) e^{-2\alpha t - \sigma^2 t} \left[ e^{2\sigma^2 t} - 2e^{\sigma^2 t} + e^{\sigma^2 t} \right]$$

$$= x^2(0) e^{-2\alpha t} \left[ e^{\sigma^2 t} - 1 \right].$$

**Q 19.** Benjamin Gompertz (1840) proposed a well-known law of mortality that had the important property that financial products based on male and female mortality could be priced from a single mortality table with an age decrement in the case of females. Cell populations are also well-known to obey Gompertzian kinetics in which $N(t)$, the population of cells at time $t$, evolves according to the ordinary differential equation

$$\frac{dN}{dt} = \alpha N \log \left( \frac{M}{N} \right),$$

where $M$ and $\alpha$ are constants in which $M$ represents the maximum resource-limited population of cells. Write down the stochastic form of this equation and deduce that $\psi = \log N$ satisfies an OU process. Further deduce that mean reversion takes place about a cell population that is smaller than $M$, and find this population.

**Answer**

The stochastic form of this SDE is

$$dN = \alpha N \log \left( \frac{M}{N} \right) dt + \sigma N dW,$$

Ito’s lemma applied to $\psi = \log N$ gives

$$d\psi = \frac{dN}{N} - \frac{\sigma^2 N^2}{2N^2} dt = \alpha \log \left( \frac{M}{N} \right) dt + \sigma dW - \frac{\sigma^2}{2} dt = \left( \alpha \log M - \alpha \psi - \frac{\sigma^2}{2} \right) dt + \sigma dW.$$

Thus $\psi$ satisfies an OU equation with mean state $\bar{\psi} = \log M - \sigma^2/2\alpha$ which in turn translates into the population $N = M e^{-\sigma^2/2\alpha}$. The standard solution of the OU equation may be used to write down the general solution for $\psi$ and consequently the general solution for $N(t)$. The result of this calculation is that

$$N(t) = \exp \left[ (1 - e^{-\alpha t}) \bar{\psi} + e^{-\alpha t} \psi_0 + \sigma \int_0^t e^{-\alpha(t-s)} dW_s \right]$$

$$= \exp \left[ (1 - e^{-\alpha t}) \log (M e^{-\sigma^2/2\alpha}) + e^{-\alpha t} \log N_0 + \sigma \int_0^t e^{-\alpha(t-s)} dW_s \right]$$

$$= (N_0 e^{-\alpha t} (M e^{-\sigma^2/2\alpha}) (1 - e^{-\alpha t})) \exp \left[ \sigma \int_0^t e^{-\alpha(t-s)} dW_s \right]$$

$$= M e^{-\sigma^2/2\alpha} \left( \frac{N_0}{M} \right)^{e^{-\alpha t}} \exp \left[ \frac{\sigma^2 e^{-\alpha t}}{2\alpha} + \sigma \int_0^t e^{-\alpha(t-s)} dW_s \right].$$
Q 20. It is given that the instantaneous rate of interest, \( r(t) \), is driven by a two-factor model in which \( r(t) = r_1(t) + r_2(t) \) where \( r_1(t) \) and \( r_2(t) \) satisfy the equations

\[
\begin{align*}
    dr_1 &= \alpha_1(\theta_1 - r_1) \, dt + \sigma_1 \sqrt{r_1} \, dW_1, \\
    dr_2 &= \alpha_2(\theta_2 - r_2) \, dt + \sigma_2 \sqrt{r_2} \, dW_2,
\end{align*}
\]

where \( dW_1 \) and \( dW_2 \) are independent increments in the Wiener processes \( W_1 \) and \( W_2 \). Let \( B(t, R_1, R_2) \) be the value of a zero-coupon bond at time \( t \) paying one dollar at maturity \( T \). Construct the partial differential equation satisfied by \( B(t, R_1, R_2) \) and state the terminal condition.

**Answer**

Ito’s lemma for \( B(t, r_1, r_2) \) gives

\[
\begin{align*}
    dB &= \frac{\partial B}{\partial t} \, dt + \frac{\partial B}{\partial r_1} \, dr_1 + \frac{\partial B}{\partial r_2} \, dr_2 + \frac{1}{2} \left( \sigma_1^2 r_1 \frac{\partial^2 B}{\partial r_1^2} dt + \sigma_2^2 r_2 \frac{\partial^2 B}{\partial r_2^2} dt \right) \\
    \Rightarrow \quad E[dB] &= \left( \frac{\partial B}{\partial t} + \alpha_1(\theta_1 - r_1) \frac{\partial B}{\partial r_1} + \alpha_2(\theta_2 - r_2) \frac{\partial B}{\partial r_2} + \frac{\sigma_1^2 r_1}{2} \frac{\partial^2 B}{\partial r_1^2} + \frac{\sigma_2^2 r_2}{2} \frac{\partial^2 B}{\partial r_2^2} \right) dt
\end{align*}
\]

But \( E[dB] = (r_1 + r_2)B \, dt \) and therefore \( B(t, r_1, r_2) \) satisfies the partial differential equation

\[
\begin{align*}
    \frac{\partial B}{\partial t} + \alpha_1(\theta_1 - r_1) \frac{\partial B}{\partial r_1} + \alpha_2(\theta_2 - r_2) \frac{\partial B}{\partial r_2} + \frac{\sigma_1^2 r_1}{2} \frac{\partial^2 B}{\partial r_1^2} + \frac{\sigma_2^2 r_2}{2} \frac{\partial^2 B}{\partial r_2^2} &= (r_1 + r_2)B.
\end{align*}
\]

Q 21. Solve the stochastic differential equation

\[
    dx_t = a(t) \, dt + b(t) \, dW_t, \quad x_0 = X_0
\]

where \( X_0 \) is constant. Show that the solution \( x_t \) is a Gaussian deviate and find its mean and variance.

**Answer**

The formal solution to this problem is

\[
    x(t) = X_0 + \int_0^t a(s) \, ds + \int_0^t b(s) \, dW_s
\]

where the second integral is to be interpreted as an Ito integral. The fact that \( x(t) \) is a Gaussian deviate stems from the fact that the stochastic part of \( x \) is simply a weighted combination of Gaussian deviates, and is therefore a Gaussian deviate. The mean is

\[
    \mu(t) = X_0 + \int_0^t a(s) \, ds
\]

and the variance is

\[
    \begin{align*}
    E[(x - \mu)^2] &= E\left[ \int_0^t b(s) \, dW_s \int_0^t b(s) \, dW_s \right] = \int_0^t b^2(s) \, ds.
\end{align*}
\]
Q 22. Solve the stochastic differential equation
\[ dx_t = -\frac{x_t \, dt}{1 + t} + \frac{dW_t}{1 + t}, \quad x_0 = 0. \]

Answer

The Ito lemma for the SDE
\[ dx_t = -\frac{x_t \, dt}{1 + t} + \frac{dW_t}{1 + t} \]
is
\[ dF = \left[ F_t(t,x) - \frac{x_t F_x(t,x)}{1 + t} + \frac{F_{xx}}{2(1 + t)^2} \right] \, dt + \frac{F_x(t,x)}{1 + t} \, dW_t \]
The idea is to choose \( F(t,x) \) so that the coefficients of \( dt \) and \( dW_t \) are functions of \( t \) alone. Clearly \( F_x(t,x) = p(t) \, x + q(t) \) is the most general expression for \( F \) for which the coefficient of \( dW_t \) is a function of \( t \) alone. With this choice, the coefficient of \( dt \) becomes
\[ F_t(t,x) - \frac{x_t F_x(t,x)}{1 + t} + \frac{F_{xx}}{2(1 + t)^2} = \frac{dp}{dt} \frac{1}{\sqrt{1 - x^2}} + \frac{dq}{dt} - \frac{p(t) \, x_t}{1 + t}. \]
Choose \( p(t) = 1 + t \) and \( q(t) = 0 \) to obtain the result
\[ d( (1 + t) \, x ) = dW_t \quad \rightarrow \quad (1 + t) \, x = W_t + x_0. \]
The final solution is therefore \( x(t) = W(t)/(1 + t) \).

Q 23. Solve the stochastic differential equation
\[ dx_t = -\frac{x_t \, dt}{2} + \sqrt{1 - x^2} \, dW_t, \quad x_0 = a \in [-1,1]. \]

Answer

The Ito lemma for the SDE
\[ dx_t = -\frac{x_t \, dt}{2} + \sqrt{1 - x^2} \, dW_t \]
is
\[ dF = \left[ F_t(t,x) - \frac{x_t F_x(t,x)}{2} + \frac{1 - x^2}{2} F_{xx} \right] \, dt + \frac{\sqrt{1 - x^2} \, F_x(t,x)}{1 + t} \, dW_t \]
Choose \( F(t,x) \) so that the coefficients of \( dt \) and \( dW_t \) are functions of \( t \) alone. This requires that
\[ F_x = \frac{p(t)}{\sqrt{1 - x^2}} \quad \rightarrow \quad F = p(t) \sin^{-1} x + q(t). \]
With this choice, the coefficient of \( dt \) becomes
\[ F_t(t,x) - \frac{x_t F_x(t,x)}{2} + \frac{1 - x^2}{2} F_{xx} = \frac{dp}{dt} \frac{1}{\sqrt{1 - x^2}} + \frac{dq}{dt} - \frac{x \, p(t)}{2\sqrt{1 - x^2}} + \frac{p(t) \, x}{2\sqrt{1 - x^2}} = \frac{dp}{dt} \frac{1}{\sqrt{1 - x^2}} + \frac{dq}{dt}. \]
Choose \( p(t) = 1 \) and \( q(t) = 0 \) to obtain the result
\[
d(\sin^{-1} x) = dW \quad \rightarrow \quad \sin^{-1} x = W_t + \sin^{-1} a .
\]
The final solution is therefore
\[
x(t) = \sin(W(t) + \sin^{-1} a) = \sqrt{1 - a^2} \sin W(t) + a \cos W(t) .
\]

Q 24. Solve the stochastic differential equation
\[
dx_t = dt + 2\sqrt{x_t} dW_t , \quad x(0) = x_0 .
\]

Answer

The Ito lemma for this SDE is
\[
dF = \left[ F_t(t,x) + F_x(t,x) + 2x F_{xx} \right] dt + 2\sqrt{x} F_x(t,x) dW_t
\]
The idea is to choose \( F(t,x) \) so that the coefficients of \( dt \) and \( dW_t \) are functions of \( t \). If \( 2\sqrt{x} F_x(t,x) = p(t) \) then \( F(t,x) = p(t) \sqrt{x} + q(t) \). The coefficient of \( dt \) is
\[
F_t(t,x) + F_x(t,x) + 2x F_{xx} = \sqrt{x} \frac{dp}{dt} + \frac{dq}{dt} + \frac{p(t)}{2\sqrt{x}} - 2x \frac{p(t)}{4x \sqrt{x}} = \sqrt{x} \frac{dp}{dt} + \frac{dq}{dt} .
\]
Choose \( p(t) = 1 \) and \( q(t) = 0 \). Thus \( F = \sqrt{x} \) and the Ito’s lemma gives
\[
dF = dW_t , \quad \rightarrow \quad \sqrt{x} = \sqrt{x_0} + W_t
\]
The solution to the SDE is therefore \( x_t = (W_t + \sqrt{x_0})^2 \).

Q 25. Solve the stochastic differential equation
\[
dx = \left[ a(t) + b(t) x \right] dt + \left[ c(t) + d(t) x \right] dW_t
\]
by making the change of variable \( y(t) = x(t) \phi(t) \) where \( \phi(t) \) is to be chosen appropriately.

Answer

\[
dy = \left[ x d\phi + (a + bx)\phi \right] dt + \left[ c(t) + d(t) x \right] \phi dW_t
\]
Choose \( \phi \) to be the solution of the equation
\[
d\phi = -\phi \left[ b(t) dt + d(t) dW_t \right]
\]
so that \( y \) satisfies
\[
 dy = \phi \left[ a(t) dt + c(t) dW_t \right] \quad \rightarrow \quad y(t) = y(0) + \int_0^t \phi(s) a(s) ds + \int_0^t \phi(s) c(s) dW_s .
\]
The determine $\phi$, we make the substitution $z = \log \phi$ to obtain
\[
dz = \left[ \frac{1}{\phi} (-\phi b(t)) - \frac{1}{2\phi^2} (d\phi^2) \right] dt + \frac{1}{\phi} (-d(\phi) \, dW_t)
\]
\[= -\left[ b(t) + \frac{d(t)^2}{2} \right] dt - d(t) \, dW_t.\]

**Q 26.** In an Ornstein-Uhlenbeck process, $x(t)$, the state of a system at time $t$, satisfies the stochastic differential equation
\[
dx = -\alpha(x - X) \, dt + \sigma \, dW_t
\]
where $\alpha$ and $\sigma$ are positive constants and $X$ is the equilibrium state of the system in the absence of system noise. Solve this SDE. Use the solution to explain why $x(t)$ is a Gaussian process, and deduce its mean and variance.

**Answer**

For any function $y = y(x,t)$, Ito’s lemma gives
\[
dy = \left[ \frac{\partial y}{\partial t} - \alpha(x - X) \frac{\partial y}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 y}{\partial x^2} \right] dt + \sigma \frac{\partial y}{\partial x} \, dW_t.
\]

Take $y = (x - X)e^{\alpha t}$ then
\[
dy = \left[ \alpha(x - X)e^{\alpha t} - \alpha(x - X)e^{\alpha t} \right] dt + \sigma e^{\alpha t} \, dW_t = \sigma e^{\alpha t} \, dW_t.
\]

Integration of this Ito equation yields
\[
(x - X)e^{\alpha t} = (x(0) - X) + \sigma \int_0^t e^{\alpha s} \, dW_s \quad \rightarrow \quad x = X + (x(0) - X)e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} \, dW_s.
\]

We observe that if $x(0)$ is constant or is itself a Gaussian deviate then $x(t)$ is simply a sum of Gaussian deviates and so is a Gaussian deviate. The mean value of $x(t)$ is
\[
\mu(t) = E \left[ X + (x(0) - X)e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} \, dW_s \right] = X + (\bar{x}(0) - X)e^{-\alpha t}.
\]

The variance of $x(t)$ is now computed from
\[
E \left[ (x - \mu)^2 \right] = E \left[ \left( (x(0) - \bar{x}(0))e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} \, dW_s \right)^2 \right]
\]
\[= E \left[ (x(0) - \bar{x}(0))^2 e^{-2\alpha t} + 2(x(0) - \bar{x}(0))e^{-\alpha t} \sigma \int_0^t e^{-\alpha(t-s)} \, dW_s \right.
\]
\[+ \sigma^2 \int_0^t e^{-\alpha(t-s)} \, dW_s \int_0^t e^{-\alpha(t-u)} \, dW_u \right]
\[= \sigma_0^2 e^{-2\alpha t} + \sigma^2 \int_0^t e^{-2\alpha(t-s)} \, ds.
\]
Q 27. Let \( x = (x_1, \ldots, x_n) \) be the solution of the system of Ito stochastic differential equations

\[
dx_k = a_k \, dt + b_{k\alpha} \, dW_{\alpha}
\]

where the repeated greek index indicates summation from \( \alpha = 1 \) to \( \alpha = m \). Show that \( x = (x_1, \ldots, x_n) \) is the solution of the Stratonovich system

\[
dx_k = \left[ a_k - \frac{1}{2} b_{j\alpha} b_{k\alpha,j} \right] \, dt + b_{k\alpha} \circ dW_{\alpha}.
\]

Let \( \phi = \phi(t, x) \) be a suitably differentiable function of \( t \) and \( x \). Show that \( \phi \) is the solution of the stochastic differential equation

\[
d\phi = \left[ \frac{\partial \phi}{\partial t} + \bar{a}_k \frac{\partial \phi}{\partial x_k} \right] \, dt + \frac{\partial \phi}{\partial x_k} b_{k\alpha} \circ dW_{\alpha}
\]

where

\[
\bar{a}_k = a_k - \frac{1}{2} b_{k\alpha,j} b_{j\alpha}.
\]

**Answer**

The SDE \( dx_i = a_i \, dt + b_{i\alpha} \, dW^\alpha \) has formal solution

\[
x_i(t) = x_i(0) + \int_0^t a_i(s, x_s) \, ds + \int_0^t b_{i\alpha}(s, x_s) \, dW^\alpha_s.
\]

The task is to relate the Ito integral in this solution to the corresponding Stratonovich integral. Each Wiener process behaves separately.

\[
\int_0^t b_{i\alpha}(s, x_s) \, dW^\alpha_s = \lim_{n \to \infty} \sum_{k=1}^n b_{i\alpha}(t_{k-1/2}, x_{k-1/2}) (W^\alpha_k - W^\alpha_{k-1})
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^n b_{i\alpha}(t_{k-1}, x_{k-1}) (W^\alpha_k - W^\alpha_{k-1})
\]

\[
+ \lim_{n \to \infty} \sum_{k=1}^n \frac{\partial b_{i\alpha}(t_{k-1}, x_{k-1})}{\partial t} (t_{k-1/2} - t_{k-1}) (W^\alpha_k - W^\alpha_{k-1})
\]

\[
+ \lim_{n \to \infty} \sum_{k=1}^n \frac{\partial b_{i\alpha}(t_{k-1}, x_{k-1})}{\partial x_j} (x^{(j)}_{k-1/2} - x^{(j)}_{k-1}) (W^\alpha_k - W^\alpha_{k-1}) + \cdots
\]

\[
= \int_0^t b_{i\alpha}(s, x_s) \, dW^\alpha_s + \lim_{n \to \infty} \sum_{k=1}^n \frac{\partial b_{i\alpha}(t_{k-1}, x_{k-1})}{\partial x_j} (x^{(j)}_{k-1/2} - x^{(j)}_{k-1}) (W^\alpha_k - W^\alpha_{k-1})
\]

It follows directly from the stochastic differential equation that

\[
x^{(j)}_{k-1/2} - x^{(j)}_{k-1} = a_j(t_{k-1}, x_{k-1}) (t_{k-1/2} - t_{k-1}) + b_j(t_{k-1}, x_{k-1}) (W^\beta_{k-1/2} - W^\beta_{k-1}) + \cdots
\]
and therefore the value of the second contribution to the Stratonovich integral is
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{\partial b_{i\alpha}(t_{k-1}, x_{k-1})}{\partial x^{(j)}} b_{j\beta}(t_{k-1}, x_{k-1}) (W_{k-1/2}^\beta - W_{k-1}^\beta) (W_k^\alpha - W_{k-1}^\alpha)
\]
\[
= \lim_{n \to \infty} \frac{1}{2} \sum_{k=1}^{n} b_{i\alpha,j}(t_{k-1}, x_{k-1}) b_{j\beta}(t_{k-1}, x_{k-1}) \delta_{\alpha,\beta} ds
\]
\[
= \frac{1}{2} \int_{t_0}^{t} b_{i\alpha,j}(t, x) b_{j\alpha}(t, x) ds.
\]
In conclusion,
\[
\int_{t_0}^{t} b_{i\alpha}(s, x_s) \circ dW_s^\alpha = \int_{t_0}^{t} b_{i\alpha}(s, x_s) dW_s^\alpha + \frac{1}{2} \int_{t_0}^{t} b_{i\alpha,j}(t, x) b_{j\alpha}(t, x) ds
\]
where repetition of $\alpha$ implies summation over the independent Wiener processes. The formal solution of the SDE with the Ito integral replaced by the Stratonovich integral is therefore
\[
x_i(t) = x_i(0) + \int_{t_0}^{t} \left( a_i(s, x_s) - \frac{1}{2} b_{i\alpha,j}(s, x_s) b_{j\alpha}(s, x_s) \right) ds + \int_{t_0}^{t} b_{i\alpha}(s, x_s) \circ dW_s^\alpha.
\]
Thus the Stratonovich form of the SDE is obtained from the Ito form by replacing the drift component $a_i$ by the modified drift $\hat{a}_i = a_i - b_{i\alpha,j} b_{j\alpha}/2$.

Let $\phi = \phi(t, x)$ then
\[
\frac{d\phi}{dt} = \partial_t \phi + \phi_j dx^{(j)} + \frac{1}{2} \phi_{ij} dx^{(i)} dx^{(j)} + \cdots
\]
\[
= \partial_t \phi + \phi_j (a_j dt + b_{j\alpha} dW_\alpha) + \frac{1}{2} \phi_{ij} b_{i\alpha} dW_\alpha b_{j\beta} dW_\beta + \cdots
\]
\[
= \partial_t \phi + \phi_j (a_j dt + b_{j\alpha} dW_\alpha) + \frac{1}{2} \phi_{ij} b_{i\alpha} b_{j\alpha} dt.
\]
Now write
\[
b_{j\alpha} dW_\alpha = b_{j\alpha} \circ dW_\alpha - \frac{1}{2} b_{j\alpha,k} b_{k\alpha} dt
\]
to obtain
\[
\frac{d\phi}{dt} = \left( \frac{\partial \phi}{\partial t} + \phi_j a_j - \frac{1}{2} \phi_{j} b_{j\alpha,k} b_{k\alpha} + \frac{1}{2} \phi_{ij} b_{i\alpha} b_{j\alpha} \right) dt + \phi_j b_{j\alpha} \circ dW_\alpha
\]
\[
= \left( \frac{\partial \phi}{\partial t} + \phi_j \hat{a}_j + \frac{1}{2} \phi_{ij} b_{i\alpha} b_{j\alpha} \right) dt + \phi_j b_{j\alpha} \circ dW_\alpha.
\]

Q 28. The displacement $x(t)$ of the harmonic oscillator of angular frequency $\omega$ satisfies $\ddot{x} = -\omega^2 x$. Let $z = \dot{x} + i\omega x$. Show that the equation for the oscillator may be rewritten
\[
\frac{dz}{dt} = i\omega z.
\]
The frequency of the oscillator randomised by the addition of white noise of standard deviation $\sigma$ to give random frequency $\omega + \sigma \xi(t)$ where $\xi \sim N(0, 1)$. Determine now the stochastic differential equation satisfied by $z$. 

\[
\frac{dz}{dt} = i\omega z.
\]
Solve this SDE under the assumption that it should be interpreted as a Stratonovich equation, and use the solution to construct expressions for

\[(a) \quad E[z(t)] \quad (b) \quad E[z(t)z(s)] \quad (c) \quad E[z(t)\bar{z}(s)].\]

**Answer**

Let \(z = \dot{x} + i\omega x\) then

\[
\frac{dz}{dt} = \frac{d^2x}{dt^2} + i\omega \frac{dx}{dt} = -\omega^2 x(t) + i\omega \frac{dx}{dt} = i\omega \left( \frac{dx}{dt} + i\omega x(t) \right) = i\omega z(t).
\]

Let \(\omega\) be replaced by \(\omega + \sigma \xi(t)\) in the equation \(\dot{z} = i\omega z\) to obtain

\[
dz = i(\omega + \sigma \xi(t))z \ dt \quad \rightarrow \quad dz = i\omega z \ dt + i\sigma z \ dW_t.
\]

We interpret this as a Stratonovich SDE and not an ITO SDE. With this interpretation, the SDE has solution

\[z(t) = z(0) \exp[i\omega t + i\sigma W(t)].\]

(a)

\[
\bar{z}(t) = E \left[ z(0) \exp[i\omega t + i\sigma W(t)] \right] = E \left[ z(0) \right] e^{i\omega t} E \left[ e^{i\sigma W(t)} \right]
\]

However, \(E \left[ e^{\sigma W(t)} \right] = e^{\sigma^2 t/2}\) and so apply this result yields the final answer

\[\bar{z}(t) = E \left[ z(0) \right] e^{i\omega t} e^{-\sigma^2 t/2} = E \left[ z(0) \right] e^{(i\omega - \sigma^2/2)t}.\]

(b)

\[
E \left[ z(t)z(s) \right] = E \left[ z(0) e^{i\omega t + i\sigma W(t)} z(0) e^{i\omega s + i\sigma W(s)} \right]
= E \left[ z(0)^2 \right] e^{i\omega (t+s)} E \left[ e^{i\sigma (W(s)+W(t))} \right].
\]

The difficulty here lies in the fact that \(W(t)\) and \(W(s)\) are correlated. Suppose \(t < s\), then \(W(t)\) and \(W(s) - W(t)\) are independent deviates. Thus

\[
E \left[ e^{i\sigma (W(s)+W(t))} \right] = E \left[ e^{2i\sigma W(t)} e^{i\sigma (W(s)-W(t))} \right]
= E \left[ e^{2i\sigma W(t)} \right] E \left[ e^{i\sigma (W(s)-W(t))} \right]
= e^{-2\sigma^2 t} e^{-\sigma^2 (s-t)/2}
= e^{-\sigma^2 (s+t)/2 - \sigma^2 t}.
\]

More generally, \(E \left[ e^{i\sigma (W(s)+W(t))} \right] = e^{-\sigma^2 (s+t)/2 - \sigma^2 \min(t,s)}\). The final result is therefore

\[E \left[ z(t)z(s) \right] = E \left[ z(0)^2 \right] e^{(i\omega - \sigma^2/2)(t+s) - \sigma^2 \min(t,s)}.
\]
(c) 
\[ E[z(t) \bar{z}(s)] = E \left[ z(0) e^{i\omega t + i\sigma W(t)} \bar{z}(0) e^{-i\omega s - i\sigma W(s)} \right] \]
\[ = E \left[ z(0) \bar{z}(0) \right] e^{i\omega (t-s)} E \left[ e^{i\sigma (W(t) - W(s))} \right] \]

Since \( W(t) - W(s) \) is a Gaussian deviate with variance \(|t-s|\), then it follows from the observation
\[ E \left[ e^{\alpha W(t)} \right] = e^{\alpha^2 t/2} \]
that
\[ E[z(t) \bar{z}(s)] = E \left[ z(0) \bar{z}(0) \right] e^{i\omega (t-s)} \sigma^2 \frac{(t-s)}{2}. \]

Q 29. It has been established in a previous example that the price \( B(t, R) \) of a zero-coupon bond at time \( t \) and spot rate \( R \) paying one dollar at maturity \( T \) satisfies the Bond Equation
\[ \frac{\partial B}{\partial t} + \mu(t, r) \frac{\partial B}{\partial r} + \frac{g(t, r)}{2} \frac{\partial^2 B}{\partial r^2} = rB \]
with terminal condition \( B(T, r) = 1 \) when the instantaneous rate of interest evolves in accordance with the stochastic differential equation \( dr = \mu(t, r) dt + \sqrt{g(t, r)} dW \).

The popular square-root process proposed by Cox, Ingersol and Ross, commonly called the CIR process, corresponds to the choices \( \mu(t, r) = \alpha(\theta - r) \) and \( g(t, r) = \sigma^2 r \). It is given that the anzatz \( B(t, r) = \exp[\beta_0(T - r) + \beta_1(T - t)r] \) is a solution of the Bond Equation in this case provided the coefficient functions \( \beta_0(T - t) \) and \( \beta_1(T - t) \) satisfy a pair of ordinary differential equations. Determine these equations with their boundary conditions.

Solve these ordinary differential equations and deduce the value of \( B(t, R) \) where \( R \) is the spot rate of interest.

**Answer**

Let \( B(t, r) = \exp[\beta_0(T - r) + \beta_1(T - t)r] \) and define \( \tau = T - t \) then
\[ \frac{\partial B}{\partial \tau} = -\left( \frac{d\beta_0}{d\tau} + r \frac{d\beta_1}{d\tau} \right) B(t, r), \quad \frac{\partial B}{\partial r} = \beta_1 B(t, r), \quad \frac{\partial^2 B}{\partial r^2} = \beta_1^2 B(t, r). \]

Substituting these calculations into the bond equation gives
\[ -r \frac{d\beta_1}{d\tau} - \alpha r \beta_1 + \frac{\sigma^2 r}{2} \beta_1^2 = r \]

which in turn leads to the equations
\[ \frac{d\beta_0}{d\tau} = \alpha \theta \beta_1 \]
\[ \frac{d\beta_1}{d\tau} = \frac{\sigma^2}{2} \beta_1^2 - \alpha \beta_1 - 1 \]

with initial conditions \( \beta_0(0) = \beta_1(0) = 0 \).

The solution of these equations is a mechanical exercise in which \( \beta_1 \) is computed first followed by \( \beta_0 \).
Q 30. In a previous exercise it has been shown that the price $B(t, R_1, R_2)$ of a zero-coupon bond at time $t$ paying one dollar at maturity $T$ satisfies the partial differential equation
\[
\frac{\partial B}{\partial t} + \alpha_1 (\theta_1 - r_1) \frac{\partial B}{\partial r_1} + \alpha_2 (\theta_2 - r_2) \frac{\partial B}{\partial r_2} + \frac{\sigma_1^2 r_1}{2} \frac{\partial^2 B}{\partial r_1^2} + \frac{\sigma_2^2 r_2}{2} \frac{\partial^2 B}{\partial r_2^2} = (r_1 + r_2) B,
\]
when the instantaneous rate of interest, $r(t)$, is driven by a two-factor model in which $r(t) = r_1(t) + r_2(t)$ in which $r_1(t)$ and $r_2(t)$ evolve stochastically in accordance with the equations
\[
\begin{align*}
dr_1 &= \alpha_1 (\theta_1 - r_1) dt + \sigma_1 \sqrt{r_1} dW_1, \\
dr_2 &= \alpha_2 (\theta_2 - r_2) dt + \sigma_2 \sqrt{r_2} dW_2,
\end{align*}
\]
where $dW_1$ and $dW_2$ are independent increments in the Wiener processes $W_1$ and $W_2$.

Given that the required solution has generic solution $B(t, r_1, r_2) = \exp[\beta_0(T - r) + \beta_1(T - t)r_1 + \beta_2(T - t)r_2]$, construct the ordinary differential equations satisfied by the coefficient functions $\beta_0$, $\beta_1$, and $\beta_2$. What are the appropriate initial conditions for these equations?

Answer

Let $B(t, r) = \exp[\beta_0(T - r) + \beta_1(T - t)r_1 + \beta_2(T - t)r_2]$ and define $\tau = T - t$ then
\[
\frac{\partial B}{\partial t} = -\left( \frac{d\beta_0}{dt} + r_1 \frac{d\beta_1}{dt} + r_2 \frac{d\beta_2}{dt} \right) B(t, r_1, r_2)
\]
\[
\frac{\partial B}{\partial r_1} = \beta_1 B(t, r_1, r_2), \quad \frac{\partial B}{\partial r_2} = \beta_2 B(t, r_1, r_2), \quad \frac{\partial^2 B}{\partial r_1^2} = \beta_1^2 B(t, r_1, r_2), \quad \frac{\partial^2 B}{\partial r_2^2} = \beta_2^2 B(t, r_1, r_2).
\]

Substituting these calculations into the bond equation gives
\[
-\left( \frac{d\beta_0}{dt} + r_1 \frac{d\beta_1}{dt} + r_2 \frac{d\beta_2}{dt} \right) + \alpha_1 (\theta_1 - r_1) \beta_1 + \alpha_2 (\theta_2 - r_2) \beta_2 + \frac{\sigma_1^2 r_1}{2} \beta_1^2 + \frac{\sigma_2^2 r_2}{2} \beta_2^2 = (r_1 + r_2),
\]
which in turn leads to the equations
\[
\begin{align*}
\frac{d\beta_0}{d\tau} &= \alpha_1 \theta_1 \beta_1 + \alpha_2 \theta_2 \beta_2, \\
\frac{d\beta_1}{d\tau} &= \frac{\sigma_1^2}{2} \beta_1^2 - \alpha_1 \beta_1 - 1, \\
\frac{d\beta_2}{d\tau} &= \frac{\sigma_2^2}{2} \beta_2^2 - \alpha_2 \beta_2 - 1,
\end{align*}
\]
with initial conditions $\beta_0(0) = \beta_1(0) = \beta_2(0) = 0$.

Q 31. The position $x(t)$ of a particle executing a uniform random walk is the solution of the stochastic differential equation
\[
dx_t = \mu \, dt + \sigma \, dW_t, \quad x(0) = X,
\]
where \( \mu \) and \( \sigma \) are constants. Find the density of \( x \) at time \( t > 0 \).

**Answer - Intuitive approach**

The SDE can be integrated immediately to get

\[
x_t = X + \int_0^t \mu(s) \, ds + \sigma W_t, \quad x(0) = X.
\]

Consequently \( x_t - X - \mu t \) is a Gaussian random deviate with mean value zero and variance \( \sigma^2 t \). Thus

\[
f(x, t) = \frac{1}{\sigma \sqrt{2\pi t}} \exp \left[ - \frac{(x - X - \mu t)^2}{2\sigma^2 t} \right].
\]

**Answer - PDE approach**

If \( x \) satisfies the SDE \( dx_t = \mu \, dt + \sigma \, dW_t \) then the density \( f(x, t) \) of \( x \) at time \( t \) satisfies the partial differential equation

\[
\frac{\partial f}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2} - \mu \frac{\partial f}{\partial x}.
\]

The task is to solve this equation with initial condition \( f(x, 0) = \delta(x - X) \). In absence of intuition, take either the Fourier transform of \( f \) with respect to \( x \) or the Laplace transform of \( f \) with respect to \( t \). For example, let

\[
\hat{f}(t; \omega) = \int_{-\infty}^{\infty} f(x, t) e^{i\omega x} \, dx
\]

then

\[
\frac{d\hat{f}}{dt} = \frac{\sigma^2}{2} \int_{-\infty}^{\infty} \frac{\partial^2 f}{\partial x^2} e^{i\omega x} \, dx - \mu \int_{-\infty}^{\infty} \frac{\partial f}{\partial x} e^{i\omega x} \, dx
\]

\[
= \left( -\frac{\omega^2 \sigma^2}{2} + \mu i \omega \right) \hat{f}
\]

with initial condition \( \hat{f}(0; \omega) = e^{i\omega X} \). Clearly the solution of this first order ODE is

\[
\hat{f}(t; \omega) = \exp \left( -\frac{\omega^2 \sigma^2 t}{2} + (X + \mu t) i \omega \right).
\]

However, this is the characteristic function of the Gaussian distribution with mean \( X + \mu t \) and variance \( \sigma^2 t \). Thus

\[
f(x, t) = \frac{1}{\sigma \sqrt{2\pi t}} e^{-\frac{(x - X - \mu t)^2}{2\sigma^2 t}}.
\]

---

**Q 32.** The position \( x(t) \) of a particle executing a uniform random walk is the solution of the stochastic differential equation

\[
dx_t = \mu(t) \, dt + \sigma(t) \, dW_t, \quad x(0) = X,
\]

where \( \mu \) and \( \sigma \) are now prescribed functions of time. Find the density of \( x \) at time \( t > 0 \).

**Answer - Intuitive approach**
This is a repeat of the previous example. The SDE can be still be integrated immediately to get

\[ x_t = X + \int_0^t \mu(s) \, ds + \int_0^t \sigma(s) \, dW_s, \quad x(0) = X. \]

In this case \( x_t - X - \int_0^t \mu(s) \, ds \) is an \( N\left(0, \int_0^t \sigma^2(s) \, ds\right) \) Gaussian random deviate. Thus

\[
\frac{1}{\sqrt{2\pi \int_0^t \sigma^2(s) \, ds}} \exp\left[-\frac{(x-X-\int_0^t \mu(s) \, ds)^2}{2\int_0^t \sigma^2(s) \, ds}\right].
\]

**Answer - PDE approach**

If \( x \) satisfies the SDE \( dx_t = \mu(t) \, dt + \sigma(t) \, dW_t \) then the density \( f(x, t) \) of \( x \) at time \( t \) satisfies the partial differential equation

\[
\frac{\partial f}{\partial t} = \frac{\sigma^2(t)}{2} \frac{\partial^2 f}{\partial x^2} - \mu(t) \frac{\partial f}{\partial x}.
\]

The task is to solve this equation with initial condition \( f(x, 0) = \delta(x - X) \). The procedure is precisely the same as the previous question, except that the Fourier transform of \( f \) is now the preferred approach. Let

\[
\hat{f}(t; \omega) = \int_{-\infty}^{\infty} f(x, t) e^{i\omega x} \, dx
\]

then

\[
\frac{d\hat{f}}{dt} = \frac{\sigma^2(t)}{2} \int_{-\infty}^{\infty} \frac{\partial^2 f}{\partial x^2} e^{i\omega x} \, dx - \mu(t) \int_{-\infty}^{\infty} \frac{\partial f}{\partial x} e^{i\omega x} \, dx
\]

\[
= \left( -\frac{\omega^2 \sigma^2(t)}{2} + \mu(t) i \omega \right) \hat{f}
\]

with initial condition \( \hat{f}(0; \omega) = e^{i\omega X} \). Clearly the solution of this first order ODE is

\[
\hat{f}(t; \omega) = \exp\left[ -\frac{\omega^2}{2} \int_0^t \sigma^2(u) \, du + \left( X + \int_0^t \mu(u) \, du \right) i \omega \right].
\]

This function is now the characteristic function of the Gaussian distribution with mean and variance

\[
X + \int_0^t \mu(u) \, du, \quad \int_0^t \sigma^2(u) \, du
\]

Thus

\[
f(x, t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\int_0^t \sigma^2(u) \, du}} \exp\left[-\frac{(x-X-\int_0^t \mu(u) \, du)^2}{2\int_0^t \sigma^2(u) \, du}\right].
\]
Q 33. The state \( x(t) \) of a particle satisfies the stochastic differential equation

\[
dx_t = a \, dt + b \, dW_t, \quad x(0) = X,
\]

where \( a \) is a constant vector of dimension \( n \), \( b \) is a constant \( n \times m \) matrix and \( dW \) is a vector of Wiener increments with \( m \times m \) covariance matrix \( Q \). Find the density of \( x \) at time \( t > 0 \).

**Answer - Intuitive approach**

This is again a repeat of the previous example. In matrix notation this SDE can be integrated immediately to get

\[
X_t = X_0 + At + BW.
\]

where \( X_t, X_0 \) and \( A \) are \( N \) dimensional column vectors, \( B \) is a constant \( N \times M \) matrix and \( W_t \) is an \( M \) dimensional column vector of correlated Wiener processes. Clearly \( X_t \) is an \( N \) dimensional Gaussian deviate with expected value \( X_0 + At \). The covariance of \( X_t \) is therefore

\[
E[(X_t - X_0 - At)(X_t - X_0 - At)^T] = E[B(W_tW_t^T)B^T] = BQB^T t = Gt
\]

where \( Q dt = E[dW_t dW_t^T] \). Thus

\[
f(X,t) = \frac{1}{(2\pi t)^{N/2}} \frac{1}{|G|^{1/2}} \exp \left[ - \frac{(X - X_0 - At) G^{-1} (X - X_0 - At)^T}{2t} \right].
\]

**Answer - PDE approach**

If \( x \) satisfies the SDE \( dx_t = a \, dt + b \, dW_t \) then the density \( f(x,t) \) of \( x \) at time \( t \) satisfies the partial differential equation

\[
\frac{\partial f}{\partial t} = \frac{g_{jk}}{2} \frac{\partial^2 f}{\partial x_j \partial x_k} - a_k \frac{\partial f}{\partial x_k},
\]

where

\[
g_{jk} = \sum_{r,s=1}^m b_{jr} Q_{rs} b_{ks}.
\]

The task is to solve this equation with initial condition \( f(x,0) = \delta(x - X) \). The \( n \)-dimensional Fourier transform of \( f \) with respect to \( x \) is defined by the formula

\[
\hat{f}(t; \omega) = \int_{R^n} f(x,t) e^{i\omega \cdot x} \, dx
\]

in which \( \omega \) is an \( n \)-dimensional vector. By taking the Fourier transform of the Kolmogorov equation satisfied by \( f(x,t) \), it follows that

\[
\frac{d\hat{f}}{dt} = \frac{g_{jk}}{2} \int_{R^n} \frac{\partial^2 f}{\partial x_j \partial x_k} e^{i\omega \cdot x} \, dx - a_k \int_{R^n} \frac{\partial f}{\partial x_k} e^{i\omega \cdot x} \, dx
\]

\[
= \left( -\frac{g_{jk} \omega_j \omega_k}{2} + a_k \omega_k i \right) \hat{f}
\]

with initial condition \( \hat{f}(0; \omega) = e^{i\omega \cdot X} \). Clearly

\[
\hat{f}(t; \omega) = \exp \left( -\frac{g_{jk} \omega_j \omega_k t}{2} + (X_k + a_k t) i \omega_k \right).
\]
By way of variety, the probability density function $f(x,t)$ is computed by direct inversion of the Fourier transform using the identity

$$f(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(t; \omega) e^{-i\omega \cdot x} d\omega$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} \left[ g_{jk} \omega_j \omega_k t + 2(x_k - X_k - a_k t) i \omega_k \right] \right) d\omega.$$ 

Let $\mathbf{v} = \mathbf{x} - \mathbf{X} - a t$, then by observing that $G = [g_{jk}]$ is a symmetric positive definite matrix, it is straightforward algebra to demonstrate that

$$g_{jk} \omega_j \omega_k t + (x_k - X_k - a_k t) i \omega_k = (\omega + i \mathbf{v} G^{-1}) G (\omega + i \mathbf{v} G^{-1})^T + \mathbf{v} G^{-1} \mathbf{v}^T.$$

Therefore,

$$f(x,t) = \exp \left[ - \frac{\mathbf{v} G^{-1} \mathbf{v}^T}{2} \right] \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[ -\frac{1}{2} (\omega + i \mathbf{v} G^{-1}) G (\omega + i \mathbf{v} G^{-1})^T \right] d\omega$$

$$= \exp \left[ - \frac{\mathbf{v} G^{-1} \mathbf{v}^T}{2} \right] \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[ -\frac{1}{2} \xi G \xi^T \right] d\xi.$$ 

In order to evaluate this integral, observe that since $G$ is symmetric and positive definite then there exists a non-singular matrix $F$ such that $G = FF^T$. Thus

$$\xi G \xi^T = \xi F F^T \xi^T = (\xi F)(\xi F)^T = \eta \eta^T,$$

$$\eta = \xi F.$$

Changing variables from $\xi$ to $\eta = \xi F$ gives

$$f(x,t) = \exp \left[ - \frac{\mathbf{v} G^{-1} \mathbf{v}^T}{2} \right] \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[ -\frac{1}{2} \xi G \xi^T \right] d\xi$$

$$= \exp \left[ - \frac{\mathbf{v} G^{-1} \mathbf{v}^T}{2} \right] \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{\det F} \exp \left[ -\frac{\eta \eta^T}{2} \right] d\eta$$

$$= \exp \left[ - \frac{\mathbf{v} G^{-1} \mathbf{v}^T}{2} \right] \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{\det F} \exp \left[ -\frac{\eta_1^2 + \eta_2^2 + \cdots + \eta_n^2}{2} \right] d\eta.$$ 

Now $|F|^2 = |G| = t^n |g|$ where $g = [g_{jk}]$. Thus $|F| = t^{n/2} |g|^{1/2}$. Furthermore,

$$\int_{\mathbb{R}^n} \exp \left[ -\frac{\eta_1^2 + \eta_2^2 + \cdots + \eta_n^2}{2} \right] d\eta = \left[ \int_{\mathbb{R}^n} \exp(-\eta^2/2) d\eta \right]^n = (2\pi)^n/2.$$ 

In conclusion, the probability density function of $\mathbf{x}$ is

$$f(\mathbf{x},t) = \frac{1}{(2\pi t)^{n/2}} \frac{1}{|g|} \exp \left[ -\frac{(\mathbf{x} - \mathbf{X} - a t) g^{-1} (\mathbf{x} - \mathbf{X} - a t)^T}{2 t} \right], \quad g = [g_{jk}].$$

Q 34. The state $x(t)$ of a system evolves in accordance with the stochastic differential equation

$$d \mathbf{x}_t = \mu \mathbf{x}_t dt + \sigma \mathbf{x}_t dW_t, \quad x(0) = X.$$
where $\mu$ and $\sigma$ are constants. Find the density of $x$ at time $t > 0$.

**Answer - Intuitive approach**

The SDE is the geometric random walk. Take $Y = \log X$ and use Ito’s lemma to construct the SDE satisfied by $Y_t$. Ito’s lemma yields

$$dY = \frac{dY}{dX} dX + \frac{\sigma^2 X^2}{2} \frac{d^2 Y}{dX^2} dt \quad \rightarrow \quad dY = \frac{1}{X} (\mu X dt + \sigma X dW_t) + \frac{\sigma^2 X^2}{2} \left( -\frac{1}{X^2} dt \right)$$

which simplifies to give

$$dY = (\mu - \sigma^2/2) dt + \sigma dW_t.$$  

This equation has solution $Y = Y_0 + (\mu - \sigma^2/2) t + \sigma W_t$. Consequently $Y$ is a Gaussian deviate with mean value $Y_0 + (\mu - \sigma^2/2) t$ and variance $\sigma^2 t$. The density of $Y$ is thus

$$f_Y(Y, t) = \frac{1}{\sigma \sqrt{2\pi t}} \exp \left[ -\frac{(Y - Y_0 - (\mu - \sigma^2/2)t)^2}{2\sigma^2 t} \right].$$

The transformation $Y = \log X$ is now used to map $f(Y, t)$ into

$$f_X(X, t) = \frac{1}{\sigma \sqrt{2\pi t} X} \exp \left[ -\frac{(\log(X/X_0) - (\mu - \sigma^2/2)t)^2}{2\sigma^2 t} \right].$$

**Answer - PDE approach**

If the state $x(t)$ of a system satisfies the stochastic differential equation $dx = \mu x dt + \sigma x dW_t$ with $\mu$ and $\sigma$ constants, then $f(x, t)$, the density of $x$ at time $t > 0$, satisfies the partial differential equation

$$\frac{\partial f(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( \sigma^2 x^2 f(x, t) \right) - \frac{\partial}{\partial x} \left( \mu x f(x, t) \right).$$

Let $z = \log x$ then

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \frac{1}{x}, \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial z^2} \frac{1}{x^2} - \frac{\partial f}{\partial z} \frac{1}{x^2}.$$

Thus it is seen that $f$ satisfies the modified PDE

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial z^2} \left( \sigma^2 x^2 f(x, t) \right) - \frac{\partial}{\partial z} \left( \mu x f(x, t) \right)$$

$$= \frac{\sigma^2}{2} \frac{\partial}{\partial x} \left( 2xf(x, t) + x^2 \frac{\partial f}{\partial x} \right) - \frac{\partial}{\partial x} \left( \mu x f(x, t) \right)$$

$$= \frac{\sigma^2}{2} \left( 2f(x, t) + x^2 \frac{\partial f}{\partial x} + x^2 \frac{\partial^2 f}{\partial x^2} \right) - \mu x \frac{\partial f}{\partial x} + f$$

$$= \frac{\sigma^2 x^2}{2} \frac{\partial^2 f}{\partial x^2} + (2\sigma^2 - \mu) x \frac{\partial f}{\partial x} + (\sigma^2 - \mu) f$$

$$= \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial z^2} + \frac{3}{2} \sigma^2 - \mu \frac{\partial f}{\partial z} + (\sigma^2 - \mu) f.$$

Now take the Fourier transform of this equation with respect to $z$ to deduce that

$$\hat{f}(t; \omega) = \int_R f(z, t) e^{i\omega z} dz$$
satisfies the ordinary differential equation

\[
\frac{df}{dt} = \left( -\frac{\sigma^2 \omega^2}{2} - i \frac{3}{2} \sigma^2 - \mu \right) \omega + (\sigma^2 - \mu)
\]

with initial condition \( \hat{f}(0, \omega) = X^{-1} e^{i \omega \log X} \). Clearly,

\[
\hat{f}(t, \omega) = X^{-1} \exp \left( -\frac{\sigma^2 \omega^2 t}{2} + i \left( \log X - \frac{3t}{2} \sigma^2 + \mu t \right) \omega + (\sigma^2 - \mu) t \right)
\]

with inverse transform

\[
f = e^{(\sigma^2 - \mu) t} \frac{1}{\sigma X \sqrt{2\pi} t} \exp \left[ - \frac{(z - \log X + \frac{3t}{2} \sigma^2 - \mu t)^2}{2\sigma^2 t} \right]
\]

\[
= e^{(\sigma^2 - \mu) t} \frac{1}{\sigma X \sqrt{2\pi} t} \exp \left[ - \frac{(\log x - \log X + \frac{3t}{2} \sigma^2 - \mu t)^2}{2\sigma^2 t} \right]
\]

\[
= e^{(\sigma^2 - \mu) t} \frac{1}{\sigma X \sqrt{2\pi} t} \exp \left[ - \frac{(\log x/X + \left( \frac{\sigma^2}{2} - \mu \right) t + \sigma^2 t)^2}{2\sigma^2 t} \right]
\]

\[
= e^{(\sigma^2 - \mu) t} \frac{1}{\sigma X \sqrt{2\pi} t} \exp \left[ - \frac{(\log x/X + \left( \frac{\sigma^2}{2} - \mu \right) t)^2}{2\sigma^2 t} - 2 \frac{\log x/X + \left( \frac{\sigma^2}{2} - \mu \right) t}{2\sigma^2 t} - 2 \frac{\sigma^2 t}{2} \right]
\]

\[
= e^{(\sigma^2 - \mu) t} \frac{1}{\sigma X \sqrt{2\pi} t} \exp \left[ - \frac{(\log x/X + \left( \frac{\sigma^2}{2} - \mu \right) t)^2}{2\sigma^2 t} - \log x/X - \left( \frac{\sigma^2}{2} - \mu \right) t - \frac{\sigma^2 t}{2} \right]
\]

\[
= \frac{1}{\sigma X \sqrt{2\pi} t} \exp \left[ - \frac{(\log x/X + \left( \frac{\sigma^2}{2} - \mu \right) t)^2}{2\sigma^2 t} - \log x/X \right]
\]

\[
= \frac{1}{\sigma X \sqrt{2\pi} t} \exp \left[ - \frac{(\log x/X + \left( \frac{\sigma^2}{2} - \mu \right) t)^2}{2\sigma^2 t} \right] \exp(- \log x/X)
\]

\[
= \frac{1}{\sigma X \sqrt{2\pi} t} \exp \left[ - \frac{(\log x/X + \left( \frac{\sigma^2}{2} - \mu \right) t)^2}{2\sigma^2 t} \right].
\]

Q 35. The state \( x(t) \) of a system evolves in accordance with the Ornstein-Uhlenbeck process

\[
dx = -\alpha \left( x - \beta \right) dt + \sigma dW_t, \quad x(0) = X,
\]

where \( \alpha, \beta \) and \( \sigma \) are constants. Find the density of \( x \) at time \( t > 0 \).

**Answer - Intuitive approach**

The SDE is reorganised into the form \( dx + \alpha (x - \beta) dt = \sigma dW_t \). Ito’s lemma is now applied to \( (x - \beta)e^{\alpha t} \) to obtain

\[
d[(x - \beta)e^{\alpha t}] = \sigma e^{\alpha t} dW_t \quad \Rightarrow \quad (x - \beta)e^{\alpha t} = (X_0 - \beta) + \sigma \int_0^t e^{\alpha s} dW_s.
\]
which simplifies to give
\[ x(t) = \beta + (X_0 - \beta)e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)}dW_s. \]

Thus \( x(t) \) is a gaussian deviate with mean value \( \beta + (X_0 - \beta)e^{-\alpha t} \) and variance
\[ \sigma^2 \int_0^t e^{-2\alpha(t-s)}ds = \frac{\sigma^2(1-e^{-\alpha t})}{2\alpha}. \]

The final conclusion is that \( x \) has probability density function
\[ f(x, t) = \frac{1}{\sigma \sqrt{2\pi(1-e^{-\alpha t})}} \exp \left[ -\frac{\alpha(x - \beta - (X_0 - \beta)e^{-\alpha t})^2}{\sigma^2(1-e^{-\alpha t})} \right]. \]

**Answer - PDE approach**

If \( x(t) \) evolves in accordance with the stochastic differential equation \( dx = -\alpha(x - \beta)dt + \sigma \, dW_t \) then the density of \( x \) at time \( t > 0 \) satisfies the partial differential equation
\[ \frac{\partial f}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2} + \alpha \frac{\partial}{\partial x} \left( (x - \beta)f \right) \]
and the initial condition \( f(x, 0) = \delta(x - X) \). Let
\[ \hat{f}(t; \omega) = \int_R f(x, t) e^{i\omega x} \, dx \]
then by taking the Fourier transform of the partial differential equation, it follows that \( \hat{f}(t; \omega) \) satisfies the ordinary differential equation
\[ \frac{\partial \hat{f}}{\partial t} = -\frac{\sigma^2}{2} \omega^2 \hat{f} + \alpha \omega \int_R \frac{\partial}{\partial x} \left( (x - \beta)f \right) e^{i\omega x} \, dx \]
and the initial condition \( \hat{f}(0, \omega) = e^{i\omega X} \). Clearly \( \phi(t, \omega) = \log \hat{f} \) satisfies the first order partial differential equation
\[ \frac{\partial \phi}{\partial t} + \alpha \omega \frac{\partial \phi}{\partial \omega} = -\frac{\sigma^2}{2} \omega^2 + i \alpha \beta \omega \]
with initial condition \( \phi(0, \omega) = i\omega X \). We may solve this equation by characteristic methods. Take \( \eta = \log \omega - \alpha t \) and \( \xi = \omega \) then
\[ \frac{\partial \phi}{\partial t} + \alpha \omega \frac{\partial \phi}{\partial \omega} = -\alpha \frac{\partial \phi}{\partial \eta} + \alpha \omega \left( \frac{\partial \phi}{\partial \xi} + \frac{1}{\omega} \frac{\partial \phi}{\partial \eta} \right) = \alpha \xi \frac{\partial \phi}{\partial \xi} = -\frac{\sigma^2}{2} \xi^2 + i \alpha \beta \xi. \]
This equation is now integrated to obtain
\[ \phi = -\frac{\sigma^2}{4\alpha} \xi^2 + i \beta \xi + \psi(\eta) \]
where the initial condition yields
\[ i\omega X = -\frac{\sigma^2\omega^2}{4\alpha} + i\beta\omega + \psi(\log \omega) \quad \rightarrow \quad \psi(\eta) = i e^\eta (X - \beta) + \frac{\sigma^2 e^{2\eta}}{4\alpha}. \]

It therefore follows that
\[
\phi = -\frac{\sigma^2\omega^2}{4\alpha} + i\beta\omega + i(X - \beta)e^{\log \omega - \alpha t} + \frac{\sigma^2}{4\alpha}e^{2(\log \omega - \alpha t)}
\]
\[
= -\frac{\sigma^2\omega^2}{4\alpha} + i\beta\omega + i(X - \beta)\omega e^{-\alpha t} + \frac{\sigma^2\omega^2}{4\alpha}e^{-2\alpha t}
\]
\[
= -\frac{\sigma^2\omega^2}{4\alpha}(1 - e^{-2\alpha t}) + i\omega(\beta + (X - \beta)e^{-\alpha t}).
\]

Thus \( \hat{f}(t; \omega) = \exp(\phi) \) corresponds to a Gaussian distribution with mean \( \beta + (X - \beta)e^{-\alpha t} \) and variance \( \sigma^2(1 - e^{-2\alpha t})/2\alpha \), that is
\[
f(x,t) = \frac{1}{\sigma} \sqrt{\frac{\alpha}{\pi(1 - e^{-2\alpha t})}} \exp \left[ -\frac{\alpha(x - \beta - (X - \beta)e^{-\alpha t})^2}{\sigma^2(1 - e^{-2\alpha t})} \right].
\]

**Q 36.** If the state of a system satisfies the stochastic differential equation
\[
dx = a(x,t)\,dt + b(x,t)\,dW, \quad x(0) = X,
\]
write down the initial value problem satisfied by \( f(x,t) \), the probability density function of \( x \) at time \( t > 0 \). Determine the initial value problem satisfied by the cumulative density function of \( x \).

**Answer**

Let \( f(x,t) \) be the probability density function corresponding to the distribution of the states of the stochastic differential equation
\[
dx = a(x,t)\,dt + b(x,t)\,dW, \quad x(0) = X,
\]
then \( f(x,t) \) satisfies the partial differential equation
\[
\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2} - \frac{\partial (af)}{\partial x}, \quad f(x,0) = \delta(x - X).
\]

Let \( F(x,t) \) be the cumulative distribution function of \( f(x,t) \) then
\[
F(x,t) = \int_{-\infty}^{x} f(u,t)\,du \quad \rightarrow \quad f(x,t) = \frac{\partial F}{\partial x}.
\]

The equation satisfied by \( F(x,t) \) is therefore
\[
\frac{\partial^2 F}{\partial x \partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( b^2 \frac{\partial F}{\partial x} \right) - \frac{\partial}{\partial x} \left( a \frac{\partial F}{\partial x} \right) \quad \rightarrow \quad \frac{\partial}{\partial x} \left[ \frac{\partial F}{\partial t} - \frac{1}{2} \frac{\partial}{\partial x} \left( b^2 \frac{\partial F}{\partial x} \right) + a \frac{\partial F}{\partial x} \right] = 0.
\]

Thus it follows that \( F \) satisfies
\[
\frac{\partial F}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x} \left( b^2 \frac{\partial F}{\partial x} \right) - a \frac{\partial F}{\partial x} + \psi(t).
\]
However, 
\[ F(x,t) \to 1, \quad \frac{\partial F}{\partial x} = f \to 0, \quad \frac{\partial^2 F}{\partial x^2} = \frac{\partial f}{\partial x} \to 0, \quad x \to \infty \]
and therefore \( \psi(t) = 0. \) In conclusion, \( F(x,t) \) satisfies the partial differential equation
\[
\frac{\partial F}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x} \left( b^2 \frac{\partial F}{\partial x} \right) - a \frac{\partial F}{\partial x} 
\]
with initial condition
\[
F(x,0) = \int_{-\infty}^{x} f(u,0) \, du = \begin{cases} 
0 & x < X, \\
1/2 & x = X, \\
1 & x > X. 
\end{cases}
\]

Q 37. Cox, Ingersoll and Ross proposed that the instantaneous interest rate \( r(t) \) should follow the stochastic differential equation
\[
dr = \alpha(\theta - r) \, dt + \sigma \sqrt{r} \, dW, \quad r(0) = r_0, 
\]
where \( dW \) is the increment of a Wiener process and \( \alpha, \theta \) and \( \sigma \) are constant parameters. Show that this equation has associated transitional probability density function
\[
f(t,r) = c \left( \frac{u}{v} \right)^{q/2} e^{-\left( \sqrt{v} - \sqrt{u} \right)^2} e^{-2\sqrt{uv}} I_q(2\sqrt{uv}),
\]
where \( I_q(x) \) is the modified Bessel function of the first kind of order \( q \) and the functions \( c, u, v \) and the parameter \( q \) are defined by
\[
c = \frac{2\alpha}{\sigma^2(1 - e^{-\alpha(t-t_0)})}, \quad u = cr_0 e^{-\alpha(t-t_0)}, \quad v = cr, \quad q = \frac{2\alpha\theta}{\sigma^2} - 1.
\]

Answer

The transitional probability density function for the stochastic differential equation
\[
dr = \alpha(\theta - r) \, dt + \sigma \sqrt{r} \, dw,
\]
satisfies the partial differential equation
\[
\frac{\partial f}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 [rf]}{\partial r^2} - \alpha \frac{\partial [(\theta - r)f]}{\partial r}, \quad f(r,0) = \delta(r - R).
\]
Since the variance is proportional to \( r \) (and cannot be negative), the sample space of \( r \) is \((0, \infty)\). Let
\[
\hat{f}(t; \omega) = \int_0^\infty f(r,t) e^{-\omega r} \, dr
\]
then the Laplace transform of the Kolmogorov equation gives

\[ \frac{\partial \hat{f}(t; \omega)}{\partial t} = \int_0^\infty \frac{\partial}{\partial r} \left[ \frac{\sigma^2}{2} \frac{\partial [rf]}{\partial r} - \alpha (\theta - r) f \right] e^{-\omega r} dr \]

\[ = \left[ \left( \frac{\sigma^2}{2} \frac{\partial [rf]}{\partial r} - \alpha (\theta - r) f \right) e^{-\omega r} \right]_0^\infty + \omega \int_0^\infty \left[ \frac{\sigma^2}{2} \frac{\partial [rf]}{\partial r} - \alpha (\theta - r) f \right] e^{-\omega r} dr \]

\[ = \omega \int_0^\infty \left[ \frac{\sigma^2}{2} \frac{\partial [rf]}{\partial r} - \alpha (\theta - r) f \right] e^{-\omega r} dr \]

\[ = \frac{\sigma^2 \omega}{2} \left[ rf e^{-\omega r} \right]_0^\infty + \frac{\sigma^2 \omega^2}{2} \int_0^\infty r f e^{-\omega r} dr - \omega \alpha \theta \hat{f}(t; \omega) + \omega \alpha \int_0^\infty r f e^{-\omega r} dr \]

\[ = -\frac{\sigma^2 \omega^2}{2} \frac{\partial \hat{f}(t; \omega)}{\partial \omega} - \omega \alpha \theta \hat{f}(t; \omega) - \omega \alpha \frac{\partial \hat{f}(t; \omega)}{\partial \omega} \]

\[ = - \left[ \frac{\sigma^2 \omega^2}{2} + \omega \alpha \right] \frac{\partial \hat{f}(t; \omega)}{\partial \omega} - \omega \alpha \theta \hat{f}(t; \omega). \]

Let \( \phi(t, \omega) = \log \hat{f}(t; \omega) \) then clearly \( \phi \) is the solution of the partial differential equation

\[ \frac{\partial \phi}{\partial t} + \left( \frac{\sigma^2 \omega^2}{2} + \omega \alpha \right) \frac{\partial \phi}{\partial \omega} = -\omega \alpha \theta \]

with initial condition \( \phi(0, \omega) = \log \hat{f}(0; \omega) = -\omega R \). Take \( \xi = \omega \) and \( \eta = \log \omega - \log(2\alpha + \sigma^2 \omega) - \alpha t \) then

\[ \frac{\partial \phi}{\partial t} + \left( \frac{\sigma^2 \omega^2}{2} + \omega \alpha \right) \frac{\partial \phi}{\partial \omega} = -\alpha \frac{\partial \phi}{\partial \eta} + \left( \frac{\sigma^2 \omega^2}{2} + \omega \alpha \right) \left( \frac{\partial \phi}{\partial \xi} + \frac{\partial \phi}{\partial \eta} \left( \frac{1}{\omega} - \frac{\sigma^2}{2\alpha + \sigma^2 \omega} \right) \right) \]

\[ = -\alpha \frac{\partial \phi}{\partial \eta} + \frac{\sigma^2 \omega^2 + 2\omega \alpha}{2} \frac{\partial \phi}{\partial \xi} + \frac{\partial \phi}{\partial \eta} \omega(2\alpha + \sigma^2 \omega) \]

\[ = \frac{\sigma^2 \omega^2 + 2\omega \alpha}{2} \frac{\partial \phi}{\partial \xi}. \]

Therefore \( \phi \) satisfies the partial differential equation

\[ \frac{\sigma^2 \omega^2 + 2\omega \alpha}{2} \frac{\partial \phi}{\partial \xi} = -\omega \alpha \theta \quad \rightarrow \quad \frac{\partial \phi}{\partial \xi} = -\frac{2\omega \alpha}{\sigma^2 \xi + 2\alpha}. \]

Thus the general solution for \( \phi \) is

\[ \phi = -\frac{2\omega \alpha}{\sigma^2} \log(\sigma^2 \xi + 2\alpha) + \psi(\eta) \]

where the initial condition gives

\[ -\omega R = -\frac{2\omega \alpha}{\sigma^2} \log(\sigma^2 \omega + 2\alpha) + \psi(\log \omega - \log(2\alpha + \sigma^2 \omega)). \]

Let \( \lambda = \log(\omega/(2\alpha + \sigma^2 \omega)) \) then

\[ \frac{\omega}{2\alpha + \sigma^2 \omega} = e^\lambda \quad \rightarrow \quad \omega = \frac{2\alpha}{e^{-\lambda} - \sigma^2}. \]

Thus

\[ \psi(\lambda) = -\frac{2\omega R}{e^{-\lambda} - \sigma^2} + \frac{2\omega \alpha}{\sigma^2} \log \left( \frac{2\alpha e^{-\lambda}}{e^{-\lambda} - \sigma^2} \right). \]
Bearing in mind that the task is to compute \( \hat{f}(t; \omega) = e^{\phi} \), it follows that

\[
\hat{f}(t; \omega) = (\sigma^2 \omega + 2\alpha)^{-(1+q)} e^{\psi(q)}
\]

\[
= (\sigma^2 \omega + 2\alpha)^{-(1+q)} \left( \frac{2\alpha}{1 - \sigma^2 e^{-\omega \alpha t}} \right)^{q+1} e^{\frac{-2\alpha R}{e^{-\omega \alpha t} - \sigma^2}}
\]

where the parameter \( q = 2\alpha \theta / \sigma^2 - 1 \). Now

\[
e^{\psi(q)} = e^{-\alpha t} \frac{\omega}{2\alpha + \sigma^2 \omega}
\]

which further simplifies \( \hat{f}(t; \omega) \) to obtain

\[
\hat{f}(t; \omega) = \left( \frac{2\alpha}{2\alpha + \sigma^2 \omega(1 - e^{-\omega t})} \right)^{q+1} e^{\frac{-2\alpha R \omega e^{-\omega t}}{2\alpha + \sigma^2 \omega(1 - e^{-\omega t})}}.
\]

Let functions \( c, u, v \) be defined by

\[
c = \frac{2\alpha}{\sigma^2(1 - e^{-\omega t})}, \quad u = cR e^{-\omega t}, \quad v = cr
\]

then

\[
\hat{f}(t; \omega) = \left( \frac{c}{c + \omega} \right)^{q+1} e^{\frac{-cR \omega e^{-\omega t}}{c + \omega}}
\]

\[
= \left( \frac{c}{c + \omega} \right)^{q+1} e^{\frac{-u \omega}{c + \omega}}
\]

\[
= \left( \frac{c}{c + \omega} \right)^{q+1} e^{-u} e^{\frac{-cu}{c + \omega}}
\]

Since \( \hat{f}(t; \omega) \) is a function of \((c + \omega)\), it follows that

\[
f(r, t) = e^{-u} e^{-cr} \mathcal{L}^{-1} \left[ (c/\omega)^{q+1} \exp(cu/\omega) : \omega \to r \right].
\]

To complete this calculation, we compute the Laplace transform of \( r^{q/2} I_q(2\sqrt{ucr}) \) where \( I_q(x) \) is the modified Bessel function of argument \( x \). The result is

\[
\mathcal{L} \left[ r^{q/2} I_q(2\sqrt{ucr}) : \omega \right] = \int_0^\infty r^{q/2} I_q(2\sqrt{ucr}) e^{-\omega r} dr
\]

\[
= \int_0^\infty r^{q/2} \left( \frac{(2\sqrt{ucr})^q}{2^q \Gamma(q + 1)} \sum_{k=0}^{\infty} \frac{(ucr)^k}{k!(q + 1)_k} \right) e^{-\omega r} dr
\]

\[
= \frac{(uc)^{q/2}}{\Gamma(q + 1)} \left( \sum_{k=0}^{\infty} \frac{(uc)^k}{k!(q + 1)_k} \right) \int_0^\infty r^{q+k} e^{-\omega r} dr
\]

\[
= \frac{(uc)^{q/2}}{\Gamma(q + 1)} \left( \sum_{k=0}^{\infty} \frac{(uc)^k}{k!(q + 1)_k} \Gamma(k + q + 1) \right)
\]

\[
= \frac{1}{c} \left( \frac{u}{c} \right)^{q/2} (c/\omega)^{q+1} \frac{\Gamma(q + 1)}{\Gamma(q + 1)} \sum_{k=0}^{\infty} \frac{(uc/\omega)^k}{k!} \Gamma(q + 1)
\]

\[
= \frac{1}{c} \left( \frac{u}{c} \right)^{q/2} (c/\omega)^{q+1} \exp(uc/\omega).
\]
Thus
\[
f(r, t) = c e^{-u} e^{-cr} \left( \frac{c}{u} \right)^{q/2} r^{q/2} I_q(2\sqrt{ucr})
\]
\[
= c e^{-u} e^{-cr} \left( \frac{cr}{u} \right)^{q/2} I_q(2\sqrt{ucr})
\]
\[
= e\left( \frac{v}{u} \right)^{q/2} e^{-\left(\sqrt{u} - \sqrt{v}\right)^2} e^{-2\sqrt{uv}} I_q(2\sqrt{uv})
\]
when \( cr \) is replaced by \( v \).

Q 38. Consider the problem of numerically integrating the stochastic differential equation
\[
dx = a(t, x) \, dt + b(t, x) \, dW, \quad x(0) = X_0.
\]
Develop an iterative scheme to integrate this equation over the interval \([0, T]\) using the Euler-Maruyama algorithm.

It is well-known that the Euler-Maruyama algorithm has strong order of convergence one half and weak order of convergence one. Explain what programming strategy one would use to demonstrate these claims.

Answer
Let \( N \) be the number of steps to be taken in advancing the solution from \( t = 0 \) to \( t = T \), then \( \Delta t = T/N \). The standard deviation of each Wiener step is therefore \( \sigma = \sqrt{\Delta} \) and the iterative scheme is captured by the pseudo-code

1. Initialize \( x \) at \( X_0 \);

2. Iterate \( N \) times \( x \rightarrow a(k\Delta t, x) \Delta t + \sigma b(k\Delta t, x) \xi \) where \( \xi \sim N(0, 1) \).

3. Final value of \( x \) is \( x(T) \)

Note that there is no requirement to store intermediate values of \( x \).

Strategy First it is necessary to simulate the integration process a large number of times, say \( M \) times, by which is meant that \( x(T) \) will be simulated \( M \) times with each simulation based on an underlying realisation of the Wiener process. Let \( x_k^{\Delta t}(T) \) be the value of \( x(T) \) returned by the \( k^{th} \) simulation when using the step size \( \Delta t \). In computing \( x_k^{\Delta t}(T) \) we choose a very fine resolution of the interval \([0, T]\), say involving \( N \) small time steps, and construct the corresponding series of \( N \) Wiener increments. These increments, appropriately packaged to build the Wiener increments \( \Delta W_{\Delta t} \) over intervals of duration \( \Delta t \), define the realisation of the underlying Wiener process needed to compute \( x_k^{\Delta t}(T) \) and, of course, by integrating at the finest resolution, i.e. with \( \Delta t = T?N \), one computes numerically ones best estimate of the true value of \( x_k(T) \) against which numerical error will be measured. Needless to say, if the SDE has an exact solution expressed in terms of \( W(T) \), then this solution would be taken as the exact solution against which error is to be estimated.
For each realisation the error $e_k = x_k^{M}(T) - x_k(T)$ is computed. To test for strong convergence one plots $\log \left( \sum_{k=1}^{M} |e_k| / M \right)$ against $\log \Delta t$ and to test for weak convergence one plots $\log \left( \sum_{k=1}^{M} e_k / M \right)$ against $\log \Delta t$. The former plot will be an approximate straight line with gradient one half and the latter plot will be an approximate straight line of gradient one.

Q 39. Compute the stationary densities for the following stochastic differential equations.

(a) $dX = (\beta - \alpha X) \, dt + \sigma \sqrt{X} \, dW$

(b) $dX = -\alpha \tan X \, dt + \sigma dW$

(c) $dX = [(\theta_1 - \theta_2) \cosh(X/2) - (\theta_1 + \theta_2) \sinh(X/2)] \cosh(X/2) \, dt + 2 \cosh(X/2) \, dW$

(d) $dX = \frac{\alpha}{X} \, dt + dW$

(e) $dX = \left( \frac{\alpha}{X} - X \right) \, dt + dW$

Answer

The stationary density $f(x)$ satisfies

$$\frac{1}{2} \frac{d(gf)}{dx} - \mu f = 0 \quad \Rightarrow \quad \frac{d(gf)}{dx} = 2\mu g f \quad \Rightarrow \quad \frac{d\log(gf)}{dx} = \frac{2\mu}{g} \quad \Rightarrow \quad f(x) = \frac{A}{g} \exp \left( \int \frac{2\mu}{g} \, dx \right),$$

where $A$ is a constant which takes the value which ensures that $f$ integrates to one.

(a) Here $\mu = (\beta - \alpha X)$ and $g(x) = \sigma^2 X$. Thus

$$f(x) = \frac{A}{\sigma^2 x} \exp \left( \int \frac{2(\beta - \alpha x)}{\sigma^2 x} \, dx \right) = \frac{A}{\sigma^2 x} \exp \left( \int \frac{2\beta}{\sigma^2 x} - \frac{2\alpha}{\sigma^2} \, dx \right) = \frac{A}{\sigma^2 x^{2\beta/\sigma^2 - 1}} e^{-2\alpha x/\sigma^2}.$$

(b) Here $\mu = -\alpha \tan X$ and $g(x) = \sigma^2$. Thus

$$f(x) = \frac{A}{\sigma^2} \exp \left( - 2 \int \frac{\tan x}{\sigma^2} \, dx \right) = \frac{A}{\sigma^2} \exp \left( 2 \log |\cos x| / \sigma^2 \right) = \frac{A}{\sigma^2} (\cos^2 x)^{1/\sigma^2}.$$

(c) Here $\mu = [(\theta_1 - \theta_2) \cosh(X/2) - (\theta_1 + \theta_2) \sinh(X/2)] \cosh(X/2)$ and $g(x) = 4 \cosh^2(X/2)$. Thus

$$f(x) = \frac{A}{4 \cosh^2(x/2)} \exp \left( \int \frac{[(\theta_1 - \theta_2) \cosh(x/2) - (\theta_1 + \theta_2) \sinh(x/2)] \cosh(x/2)}{4 \cosh^2(x/2)} \, dx \right)$$

$$= \frac{A}{4 \cosh^2(x/2)} \exp \left( \int \frac{\theta_1 - \theta_2}{4} - \frac{\theta_1 + \theta_2 \sinh(x/2)}{4 \cosh(x/2)} \, dx \right)$$

$$= \frac{A}{4 \cosh^2(x/2)} \exp \left( \int \frac{\theta_1 - \theta_2}{4} - \frac{\theta_1 + \theta_2}{2} \log \cosh(x/2) \right)$$

$$= \frac{A}{4 \cosh^{2+(\theta_1+\theta_2)/2}(x/2)} \exp \left( \frac{(\theta_1 - \theta_2)}{4} \right).$$
(d) Here $\mu = \alpha/X$ and $g(x) = \sigma^2$. Thus

$$f(x) = \frac{A}{\sigma^2} \exp\left( \int \frac{2\alpha}{\sigma^2 x} \, dx \right) = \frac{A}{\sigma^2} \exp\left( \frac{2\alpha}{\sigma^2} \log x \right) = \frac{A}{\sigma^2} x^{2\alpha/\sigma^2}.$$

(e) Here $\mu = (\alpha/X - X)$ and $g(x) = \sigma^2$. Thus

$$f(x) = \frac{A}{\sigma^2} \exp\left( 2 \int \frac{\alpha/x - x}{\sigma^2} \, dx \right) = \frac{A}{\sigma^2} \exp\left( \frac{2\alpha}{\sigma^2} \log x - \frac{x^2}{\sigma^2} \right) = \frac{A}{\sigma^2} x^{2\alpha/\sigma^2} e^{-x^2/\sigma^2}.$$