3

Static Games of Complete Information: Pure Strategy NE in Finite Games

Kerschbamer: Commitment and Information in Games

Pure Strategy Nash Equilibrium in Finite Games

We now turn to the **central solution concept** in game theory, that of a **Nash equilibrium (NE)**.

For ease of exposition, we initially concentrate on **pure strategies**, restricting attention to games of the form $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$. Mixed strategy equilibria are studied in the next lecture.

Also, we first look at **finite games**. Games with continuous strategy spaces are studied in Lecture 5.

Informal Definition of Nash Equilibrium: Suppose the players in a game ask a game theorist how to play the game. Further suppose the game theorist makes an unique recommendation about the strategy each player should choose. The resulting strategy profile is a **Nash equilibrium**, if each player is willing to choose the strategy recommended to her by the theorist given she expects that the opponents follow the recommendation too

Definition of Nash Equilibrium

Here is a formal version of the definition:

Definition 12a (formal version 1): A strategy profile $s^* = (s^*_1, \ldots, s^*_n) \in S$ constitutes a **Nash equilibrium** of game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ if for each $i \in N$ we have

$$u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i}) \qquad \forall\ s_i \in S_i.$$

Note: At s^* **no** player has an **incentive to deviate** from her part in s^*, **given** she expects her **opponents** to **follow their part in s^*,** too.

Also note: **Only** (Nash) **equilibrium strategy profiles have** this **self-enforcing property** (in the sense that no single player wants to deviate from her predicted strategy, given she expects the rivals to follow the recommendation too).

Therefore: It would be **self-defeating** for a game theorist to give a **recommendation** for each player that was **not a Nash equilibrium**. If the advice were adopted by all players then at least one player would "regret" ex post having carried out the theorist's advice.

Finding Nash Equilibria

In finite games a **brute-force approach** to finding a game's Nash equilibria is simply to check whether each possible strategy profile (each cell in the bi-matrix form) satisfies the condition in Definition 12.

Try the following game:

s_1 \ s_2	**L**	**C**	**R**
U	2, 0	1, 1	4, 2
M	3, 2	1, 2	2, 3
D	1, 2	0, 2	3, 0

We see: For finite games the **brute-force approach** works, but it **is** quite **burdensome**, especially for large (finite) games.

Finding Nash Equilibria (Cont.)

A **more systematic approach** to finding a game's Nash equilibria begins as follows:

For each player and for each strategy of that player, **underline** the other player's **best-response** to that strategy:

s_2 s_1	**L**	**C**	**R**
U	2, 0	$\underline{1}$, 1	$\underline{4}$, $\underline{2}$
M	$\underline{3}$, 2	$\underline{1}$, 2	2, $\underline{3}$
D	1, $\underline{2}$	0, $\underline{2}$	3, 0

A pair of strategies satisfies the condition for a **Nash equilibrium** in Definition 12 if each player's strategy is a best-response to the other's – that is, if **both payoffs** are **underlined** in the corresponding cell of the bi-matrix.

Best-Response Correspondences and Nash Equilibria

The more systematic approach of the previous slide suggests an **alternative** (equivalent) **definition for Nash equilibrium** involving best-responses.

Definition 13: The **best-response correspondence** for player $i \in N$ in $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ is a set-valued function B_i such that:

$$B_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}), \ \forall \ s'_i \in S_i \}.$$

Note: $B_i(s_{-i}) \subseteq S_i$ "tells" player i what to do when the other players play s_{-i}.

Also Note: $B_i(s_{-i})$ might contain more than one element (if the player has several - equally good - best-responses).

Definition 12b (formal version 2). A strategy profile $s^* = (s^*_1, \ldots, s^*_n) \in S$ constitutes a **Nash equilibrium** of game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ if and only if $s^*_i \in B_i(s^*_{-i})$ for each $i \in N$.

In words: A Nash equilibrium is a strategy profile of mutual best-responses.

Applying the Alternative Definition

Example 5 again

s_2 / s_1	**L L L**	**L L R**	**L R L**	**L R R**	**R L L**	**R L R**	**R R L**	**R R R**
l l	5, 0	5, 0	$\underline{5}$, 0	5, 0	1, 4	1, 4	1, 4	1, 4
l r	5, 0	5, 0	$\underline{5}$, 0	5, 0	1, 4	1, 4	1, 4	1, 4
r l	$\underline{6}$, 3	$\underline{6}$, 3	4, 5	6, 2	6, 3	6, 3	4, 5	6, 2
r r	$\underline{6}$, 3	$\underline{6}$, 3	2, 1	6, 2	6, 3	6, 3	2, 1	6, 2

$B_1(LLL) = \{rl, rr\}$
$B_1(LRL) = \{ll, lr\}$
$B_1(RLL) = \{\ldots, \ldots\}$
$B_1(RRL) = \{\ldots, \ldots\}$

$B_1(LLR) = \{rl, rr\}$
$B_1(LRR) = \{\ldots, \ldots\}$
$B_1(RLR) = \{\ldots, \ldots\}$
$B_1(RRR) = \{\ldots, \ldots\}$

$B_2(ll) = \{\ldots, \ldots\}$
$B_2(lr) = \{\ldots, \ldots\}$
$B_2(rl) = \{\ldots, \ldots\}$
$B_2(rr) = \{\ldots, \ldots\}$

Nash Equilibrium: strategy profile $s^* = (s^*_1, s^*_2)$ where $s^*_1 \in B_1(s^*_2)$ and $s^*_2 \in B_2(s^*_1)$.

Example shows: There may be **many Nash equilibria** in a given game (more on this below).

Motivating Nash Equilibrium

A number of arguments have been put forward to **motivate the concept of Nash Equilibrium (NE):**

(i) NE as a necessary condition if there is an unique predicted outcome to a game: "If there is an unique predicted outcome for a game, rational player will understand this. Therefore, for no player to wish to deviate, this predicted outcome must be a NE."

⇨ argument only relevant if there is an unique predicted outcome

(ii) NE as a focal point: "Suppose one of the outcomes in a game has a 'natural appeal'. Then this outcome is a natural prediction in that game if the outcome is a NE."

⇨ argument convincing in some contexts; but no guarantee that outcome with ‘n.a’ exists

Motivating Nash Equilibrium (Cont.)

(iii) **NE as a self enforcing agreement**: "Imagine that the players can engage in nonbinding communication prior to playing a game. If players reach an agreement how to play the game, this naturally becomes the obvious candidate for play. However, since the agreement is non-binding, it must be self-enforcing (and therefore it must involve playing NE strategies) to be meaningful."

⇨ argument assumes that the strategies agreed upon become focal

(iv) **NE as a stable social convention or as a rest point of some dynamic adjustment process**: "Suppose the game is played repeatedly against randomly selected opponents. Also suppose that some stable convention emerges. Then this convention must be a NE. If it were not, individuals would deviate from it as soon as it began to emerge."

⇨ if players are perfectly rational: repeated one-shot games should be viewed as dynamic games; why should we expect NE in this larger game?

⇨ if players are not perfectly rational: rules of thumb (e.g., myopic best-response) do not always converge to a stable outcome; thus, argument relevant in some contexts but not in others

Relation between Nash Equilibrium and Iterated Deletion

What is the relation between **Nash Equilibrium (NE)** and **iterated deletion of strictly dominated strategies (ID)**?

Here is the answer: **NE is** a **stronger** solution concept **than ID** in the following sense:

Proposition 2. If the strategy profile $s^* = (s^*_1,\ldots,s^*_n)$ is a Nash equilibrium in the game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ then this strategy profile survives iterated deletion of strictly dominated strategies.

Proof. On next slide. ■

Note: The **converse** is **not true;** that is, there can be strategy profiles that survive ID but are not part of any NE (show in earlier examples). On the other hand, **if only one** strategy **profile survives ID** this strategy **profile is** a **NE** (and it is the only one):

Proposition 3. If iterated deletion of strictly dominated strategies eliminates all but the strategy profile $s^* = (s^*_1,\ldots,s^*_n)$ in the normal form game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, then this strategy profile is the unique Nash equilibrium of the game.

Proof. On one of the next slides. ■

To sum up: The concept of a **NE offers at least as sharp a prediction as does ID**. In fact, **it often offers a much sharper prediction**.

Relation between Nash Equilibrium and Iterated Deletion (Cont.)

Proof of Proposition 2. The proof is **by contradiction**. That is, we start by assuming that one of the strategies in a NE is eliminated by ID and then show that this results in a contradiction (proving that the assumption must be wrong).

- Suppose $s^* = (s^*_1, \ldots, s^*_n)$ is a NE in $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$.
- Further suppose that s^*_i is the first strategy in s^* to be eliminated by ID.
- Then there must exist a strategy s'_i and (using the notation introduced in Definition 11b) some t such that

$$s'_i \neq s^*_i \in A^t_i \text{ and}$$

$$u_i(s^*_i, s_{-i}) < u_i(s'_i, s_{-i}) \ \forall \ s_{-i} \in A^t_{-i} \tag{a}$$

- Now since s^*_i is the first strategy in s^* to be eliminated, we must have $s^*_{-i} \in A^t_{-i}$.
- Thus, one of the inequalities in (a) is

$$u_i(s^*_i, s^*_{-i}) < u_i(s'_i, s^*_{-i})$$

- But then $s^* = (s^*_i, s^*_{-i})$ is not a NE. ■

Relation between Nash Equilibrium and Iterated Deletion (Cont. 2)

Proof of Proposition 3. The fact that, if the strategy profile s^* is a NE, then it is unique, follows from Proposition 2. Thus, all we need to show is that if ID yields an unique profile s^*, then this profile is a NE. The argument is again **by contradiction**:

- Suppose that ID yields s^* as the unique strategy profile, but s^* is not a NE.
- Then (since s^* is not NE) there exists some i and some s_i in S_i such that

$$u_i(s^*_i, s^*_{-i}) < u_i(s_i, s^*_{-i}) \qquad \text{(b)}$$

- Also (since s_i does not survive ID), there exists some s'_i and (again using the notation introduced in Definition 11b) some t such that $s'_i \neq s_i \in A^t_i$ and

$$u_i(s_i, s_{-i}) < u_i(s'_i, s_{-i}) \ \forall\ s_{-i} \in A^t_{-i} \qquad \text{(c)}$$

- Since s^*_{-i} is never eliminated by assumption, (c) implies

$$u_i(s_i, s^*_{-i}) < u_i(s'_i, s^*_{-i}) \qquad \text{(d)}$$

- If $s'_i = s^*_i$ then (d) contradicts (b) and we are done.
- If $s'_i \neq s^*_i$ then some other strategy s''_i must later strictly dominate s'_i (since s'_i does not survive ID)
- Thus, inequalities analogous to (c) and (d) hold with s'_i and s''_i replacing s_i and s'_i, respectively.
- Once again, if $s'' = s^*_i$ then we are done.
- If $s'' \neq s^*_i$, repeat the step above until the new strategy is s^*_i (since s^*_i is the only strategy in S_i to survive ID, the process must stop after finitely many steps in any finite game). ■

Relation between NE and Equilibrium in Strictly Dominant Strategies

Obviously, **equilibrium in strictly dominant strategies** is **stronger than NE** in the following sense:

Proposition 4. If the strategy profile $s^* = (s^*_1, \ldots, s^*_n)$ is **an equilibrium in strictly dominant strategies** in the normal form game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, then this strategy profile **is the unique Nash equilibrium** of the game. The **converse is not true**, of course. That is, a strategy profile may be a (or the unique) Nash equilibrium without being an equilibrium in strictly dominant strategies.

Proof: Left as an exercise in Problem Set 3 ■

Note: **Equilibrium in strictly dominant strategies** is the **most plausible solution concept**. The motivation for looking for other solution concepts is precisely the fact that equilibrium in strictly dominant strategies is **too strong** a solution concept (in the sense that it rarely exists!)

Existence of Nash Equilibrium (in Pure Strategies)

Having shown that NE is a stronger solution concept than ID, we must now ask whether NE is too strong a solution concept. That is, **can we be sure that a NE always exists**?

Example 1, again

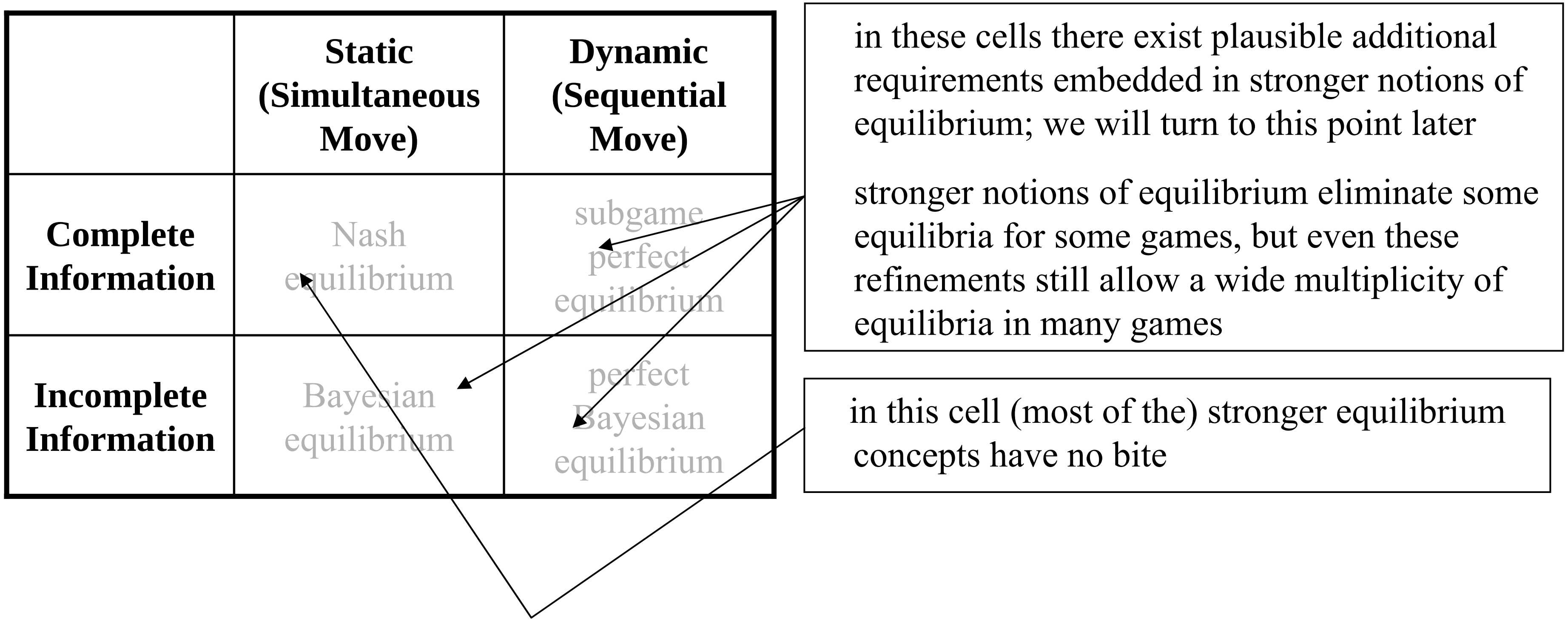
child 1 \ child 2	Rock	Paper	Scissors
Rock	**0, 0**	**-1, 1**	**1, -1**
Paper	**1, -1**	**0, 0**	**-1, 1**
Scissors	**-1, 1**	**1, -1**	**0, 0**

Note: There is **no NE in pure strategies** to this game. However, there is **one in mixed strategies** (more on this in the next lecture). We will show later: **NE** (not necessarily in pure strategies) **exist under fairly broad circumstances!** In particular, **any finite game has at least one NE** (possibly involving mixed strategies). More on this in the next lecture.

Multiplicity of Nash Equilibria

- on previous slide we have argued that **NE exists in finite games** (see Lecture 4 for the formal result)
- we will see later (in Lecture 5) that **NE exist** under fairly broad circumstances **even in infinite games**
- so existence is not really a problem, the real **problem with NE** is that many games have **more than one NE**
- indeed, it is easy to construct games in which the set of NE is quite large (see, e.g. Example 5)
- if a game has more than one NE, **which** of the **NE** should be **selected?**
- in other words, what makes **one NE more plausible than** the **other ones?**
- the answer to this question depends on the game being played
- **for many games there exist** (more or less) compelling criteria to select a unique element or a subset of the set of NE (see next slide)
- those criteria are embedded in **more refined** (and therefore stronger) **notions of equilibrium**
- e.g., a subgame perfect equilibrium (to be studied in l. 6, 7 and 8) is a NE which fulfils an additional requirement (that players should not be allowed to make incredible threats or incredible promises)

Multiplicity of Nash Equilibria (Cont. 1)



	Static (Simultaneous Move)	**Dynamic (Sequential Move)**
Complete Information	Nash equilibrium	subgame perfect equilibrium
Incomplete Information	Bayesian equilibrium	perfect Bayesian equilibrium

in these cells there exist plausible additional requirements embedded in stronger notions of equilibrium; we will turn to this point later

stronger notions of equilibrium eliminate some equilibria for some games, but even these refinements still allow a wide multiplicity of equilibria in many games

in this cell (most of the) stronger equilibrium concepts have no bite

- so the question remains: if a game has more than one equilibrium, which of these, if any, should we expect to be played?
- we know that any of these equilibria, if it were expected by all players, could become a self-fulfilling prophecy
- so, the relevant question to be asked is, what might cause the players in a game to expect each other to play a particular equilibrium?

Multiple Equilibria and Focal Points

- in a game with multiple equilibria, anything that tends to focus the players' attention on some equilibrium (the **focal equilibrium**) may make them all expect it an hence fulfil it (Schelling, 1960)
- that is, a **focal equilibrium** is an equilibrium that has some property that distinguishes it from all the other equilibria
- but, **what makes an equilibrium a focal equilibrium**?
  - in some games there are many equilibria but there is a unique **symmetric equilibrium** it is then often argued that this is enough to guarantee focal status for this equilibrium (show example)
  - in other games, one equilibrium **Pareto-dominates** the other equilibria; again, it is then often argued that this is enough to guarantee focal status for this equilibrium (show example)
  - in still other games, one equilibrium **risk-dominates** the other equilibria; ... (show example)
  - also, **equity considerations** may emphasis one equilibrium; ... (show example)
  - there are many **other factors** that might lead to a focal equilibrium

Multiple Equilibria and No Way To Select

- what if **different factors** tend to **focus on different equilibria**? For instance:

s_1 \ s_2	**L**	**R**
U	10, 10	0, x
D	x, 0	x, x

- suppose $x \in [0, 10]$
- then the game has two pure strategy NE
- if $x < 10$ then one equilibrium Pareto dominates the other, should it therefore always be selected?
- what if $x = 9$; or 9.9; or 9.99?

- and what if **no focal point** is apparent **in** the **game itself**?
- then, to **locate** a **focal point**, it might be necessary to look for clues, not in the game itself, but **in** the **real-world situation** from which the game has been abstracted (Binmore 1982, pp. 296)
- human societies abound with **conventions**, or **cultural norms**, that exist precisely for equilibrium selection purposes ("driving on the right", "doing what the wife wants", etc.)
- from a game-theoretic perspective, cultural norms can be defined to be the rules a society uses to determine focal equilibria in game situations (Myerson 1991, p. 113)

Or is there Always a Way to Select?

- this is a controversial question
- we will not pursue it further here
- for those who are interested in this kind of question a useful **starting point** remains the original treatment by

> Schelling, T. (1960), *The Strategy of Conflict*, Cambridge, Mass.: Harvard University Press

- a book focusing on **"Pareto – dominance"** and **"risk – dominance"** and the question which of these two criteria should be used when is

> Harsanyi, J. and R. Selten (1988), *A General Theory of Equilibrium Selection in Games*, Cambridge, Mass.: MIT Press