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Static Games of Complete Info: Dominance

Kerschbamer: Commitment and Information in Games

Static Games of Complete Information

Having discussed how to translate an informal description of an interactive decision in a game, the next step is to analyze the game, to **predict what will (and will not) happen** if the game is actually played by rational players who are fully knowledgeable about the structure of the game and each others rationality.

In non-cooperative game theory two so-called solution techniques are generally used to come to a prediction, dominance concepts and equilibrium concepts.

This lecture is on **dominance**. The rest of the semester we will deal with **equilibrium** only occasionally turning to dominance.

In this lecture we concentrate on static games of complete information - we discuss later how the concepts can be adapted to be of help in other games.

A **static game** (or **simultaneous move game**) is a game in which all players move only once and at the same time.

A **game of complete information** is a game in which all players know all relevant information about each other, including the payoffs that each receives from the various outcomes of the game.

Strictly Dominant Strategies

We begin with the most compelling behavioural prediction

Initially we ignore the possibility that players might randomize in their strategy choice, we turn to this issue later.

Definition 7a: A strategy $s_i \in S_i$ is a **strictly dominant strategy** for player i in game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ if for all $s'_i \neq s_i \in S_i$ we have $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ for all $s_{-i} \in S_{-i} = \times_{j \neq i} S_j$ (replace " > " by "≥" and add "with strict inequality for some s_{-i}" to get the definition of a **weakly dominant strategy**.)

In words: A strategy s_i is a strictly dominant strategy for player i if it is strictly better than any other available strategy, no matter what the opponents play.

Of course, if a rational player has a strictly dominant strategy **she should choose it**. No information about other players' rationality and knowledge of the game is required, own rationality is enough!

Definition 8: The strategy profile $s^* \in S$ is an **equilibrium in strictly dominant strategies** in game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ if s^*_i is a strictly dominant strategy for each player $i \in N$.

Applying the Concept of Strict Dominance - Example 9: Prisoners' Dilemma

- two (male) individuals are arrested and charged with a crime
- the attorney (she) lacks sufficient evidence to convict the suspects, unless at least one confesses ("defects")
- she puts the suspects in separate cells and tells each of them in private that
  - if he is the only one to defect (D), he is freed, while the second suspect is imprisoned for a very long time
  - if both defect, then both will be sentenced for a medium time
  - if neither defects, then both will be convicted for a minor offence and sentenced for a short time
- both suspects are only interested on any jail term they individually serve (long jail is bad)

The **story** above **implies** the following relations between payoffs: c > a > d > b

- what will the outcome of this game be?
- there is only one plausible answer: …
- which outcomes are Pareto optimal?

example shows: self-interested, individually rational behaviour might not lead to a socially optimal result.
applications?

	C	D
C	a, a	b, c
D	c, b	d, d

Strictly Dominated Strategies

The notion that (rational) *players should play* their *strictly dominant strategies* if they have them is compelling. However, *strictly dominant strategies rarely exist*. Often, one strategy is best for player i if opponents play s_{-i} and another strategy is best if opponents play s'_{-i}. Still the idea of dominance may help to eliminate some strategies as possible choices.

Definition 9a (pure strategy version): Strategy $s_i \in S_i$ **strictly dominates** strategy $s'_i \neq s_i \in S_i$ for player i in game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ if $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ for all $s_{-i} \in S_{-i} = \times_{j \neq i} S_j$ (again, replace " > " by "≥" and add "with strict inequality for some s_{-i}" to get the definition of the "**weakly dominates**" relation).

Definition 10a (pure strategy version): Strategy $s'_i \in S_i$ is **strictly dominated** in game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ if there is another strategy $s_i \neq s'_i \in S_i$ that strictly dominates it.

Of course, if a rational player has a strictly dominated strategy *she should not choose it* as there is another strategy that yields her a greater payoff regardless of what the opponents do.

Again, no info about other players' rationality and knowledge of the game is required.
Again, own rationality is sufficient.

Strictly Dominated Strategies (Cont.)

Definition 9a allows us to restate our definition of a strictly dominant strategy as follows:

Definition 7b: A strategy $s_i \in S_i$ is a **strictly dominant strategy** for player i in game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ if it strictly dominates all $s'_i \neq s_i \in S_i$.

Example 10

s_1 \ s_2	**L**	**M**	**R**
U	1, 0	1, 2	0, 1
D	0, 3	0, 1	2, 0

There is no strictly dominant strategy in this game.
Is there a strictly dominated strategy?

Iterated Deletion of Strictly Dominated Strategies

The notion that (rational) *players should not play* a *strictly dominated strategy* is compelling. However, the *elimination of strictly dominated strategies rarely leads to* an *unique prediction* for a game. Sometimes pushing the logic of eliminating strictly dominated strategies further is of some help:

Definition 11a (informal version): Strategy $s_i \in S_i$ survives **iterated deletion of strictly dominated strategies** in game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ if it survives the following process: All players simultaneously eliminate all strictly dominated strategies. If at least one of the players has one strictly dominated strategy left in the reduced game, the elimination is repeated until there is no strictly dominated strategy left.

Example 10 again

s_1 \ s_2	**L**	**M**	**R**
U	1, 0	1, 2	0, 1
D	0, 3	0, 1	2, 0

Practicing Iterated Deletion

Consider example 10 and let μ stand for "*rational and knows the game*".

For the *column player* (*cp*, she), *R* is strictly dominated by *M*. Thus, if the *cp* is μ then she will not play *R*. If the *row player* (*rp*, he) knows that the *cp* is μ, then he will play as if the game were

	L	M
U	1, 0	1, 2
D	0, 3	0, 1

In this game, for the *rp D* is strictly dominated by *U*. Thus, if *rp* is μ and if he knows that *cp* is μ, then he will not play *D*. If *cp* knows that *rp* is μ and that *rp* knows that she (the *cp*) is μ, then she will play as if the game were

	L	M
U	1, 0	1, 2

In this game, for *cp L* is strictly dominated by *M*. Thus, if *cp* is μ and if she knows that *rp* is μ and that *rp* knows that she (the *cp*) is μ, then she will not play *L*.

Thus the unique strategy profile surviving iterated deletion of strictly dominated strategies in Example 10 is $(s_1, s_2) = (U, M)$

Iterated Deletion, Rationality and Common Knowledge

Elimination of strictly dominated strategies requires only that each player is rational. In particular, no information about opponents' payoffs or about their rationality is required for a player to eliminate a strictly dominated strategy from consideration as her own strategy choice.

Iterated deletion of strictly dominated strategies is much more demanding. To eliminate strategies that are strictly dominated after the first deletion of strategies players must not only be rational, they must also know that their opponents are rational. And each additional iteration requires that players' knowledge of each others' rationality be one level deeper. A player must now know not only that her opponents are rational but also that they know that she is, and so on.

If we want to be able to apply the process of iterated deletion for an arbitrary number of steps, we need to assume that it is **common knowledge** that the players are rational. That is, we need to assume not only that all the players are rational, but also that all the players know that all the players are rational, and that all players know that all players know that all the players are rational, and so on, *ad infinitum*.

On the positive side, the **order of deletion** does not affect the set of strategies that remain in the end (see next page).

Formally Deleting Strictly Dominated Strategies

Definition 11b (formal version): Strategy profiles in $A \subseteq S$ **survive iterated deletion of strictly dominated strategies** in game $G^N = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ if $A = \times_{i \in N} A_i$ and there is a collection $\{\{A^t_i\}_{i \in N}\}^T_{t=0}$ of sets such that, for all $i \in N$:

1. $A^0_i = S_i$ and $A^T_i = A_i$.
2. $A^{t+1}_i \subseteq A^t_i$ for all $t \in \{0, \ldots, T-1\}$.
3. For all $t \in \{0, \ldots, T-1\}$, every strategy of player i, $a_i \in A^t_i \setminus A^{t+1}_i$ is strictly dominated in the game $[N, \{A^t_j\}_{j \in N}, \{u^t_j\}_{j \in N}]$, where u^t_j for all $j \in N$ is the function u_j restricted to $\times_{i \in N} A^t_i$.
4. No $a_i \in A^T_i$ is strictly dominated in the game $[N, \{A^t_j\}_{j \in N}, \{u^T_j\}_{j \in N}]$.

Note: A is non-empty, and does not depend upon the strategy-deletion order.

In sum: It is easier to apply iterated deletion of strictly dominated strategies than it is to define it formally!

Mixed Strategies and Dominance

When we recognize that players may randomize over their pure strategies, Definition 9a and Definition 10a can be generalized in a straightforward way:

Definition 9b (mixed strategy version): Strategy $\sigma_i \in \Delta(S_i)$ **strictly dominates** strategy $\sigma'_i \neq \sigma_i \in \Delta(S_i)$ for player i in game $\Gamma^N = [N, \{\Delta(S_i)\}_{i \in N}, \{U_i\}_{i \in N}]$, if $U_i(\sigma_i, \sigma_{-i}) > U_i(\sigma'_i, \sigma_{-i})$ for all $\sigma_{-i} \in \Delta(S_{-i}) = \times_{j \neq i} \Delta(S_j)$.

Definition 10b (mixed strategy version): Strategy $\sigma'_i \in \Delta(S_i)$ is **strictly dominated** in game $\Gamma^N = [N, \{\Delta(S_i)\}_{i \in N}, \{U_i\}_{i \in N}]$ if there is another strategy $\sigma_i \neq \sigma'_i \in \Delta(S_i)$ that strictly dominates it.

Proposition 1: Player i's pure strategy $s'_i \in S_i$ is strictly dominated in the game $\Gamma^N = [N, \{\Delta(S_i)\}_{i \in N}, \{U_i\}_{i \in N}]$ if and only if there exists another strategy $\sigma_i \in \Delta(S_i)$ such that $U_i(\sigma_i, s_{-i}) > U_i(s'_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

Note: Proposition 1 slightly abuses notation: U_i's domain are mixed strategy profiles. What does $U_i(\sigma_i, s_{-i})$ mean then?

Also note: Proposition 1 tells us that to test whether a **pure strategy s_i is dominated** when mixed strategies are allowed, the **test** given in Definition 9a needs only be augmented by checking **whether any of** player ***i*'s mixed strategies does better than s_i against** any possible profiles of **pure strategies by** *i*'s **opponents**.

Mixed Strategies and Dominance (Cont.)

Proof of Proposition 1: We show that when we test whether a strategy σ'_i is strictly dominated by strategy σ_i for player i, we need only consider these two strategies' payoffs against the **pure** strategies of i's opponents. That is, $U_i(\sigma_i, \sigma_{-i}) > U_i(\sigma'_i, \sigma_{-i})$ for all σ_{-i} if and only if $U_i(\sigma_i, s_{-i}) > U_i(\sigma'_i, s_{-i})$ for all s_{-i} . This follows because

$$U_i(\sigma_i,\sigma_{-i}) - U_i(\sigma'_i,\sigma_{-i}) = \sum_{s_{-i}\in S_{-i}} \left[\prod_{k\neq i}\sigma_k(s_k)\right]\left[U_i(\sigma_i,s_{-i}) - U_i(\sigma'_i,s_{-i})\right].$$

This expression is positive for all σ_{-i} if and only if $[U_i(\sigma_i, s_{-i}) - U_i(\sigma'_i, s_{-i})]$ is positive for all s_{-i}. ■

Note: A **pure strategy** s_i may be **dominated only by** a **mixed strategy**! That is, do dominate a strategy (even a pure one) it may be necessary to consider mixed strategies (see example on next slide)

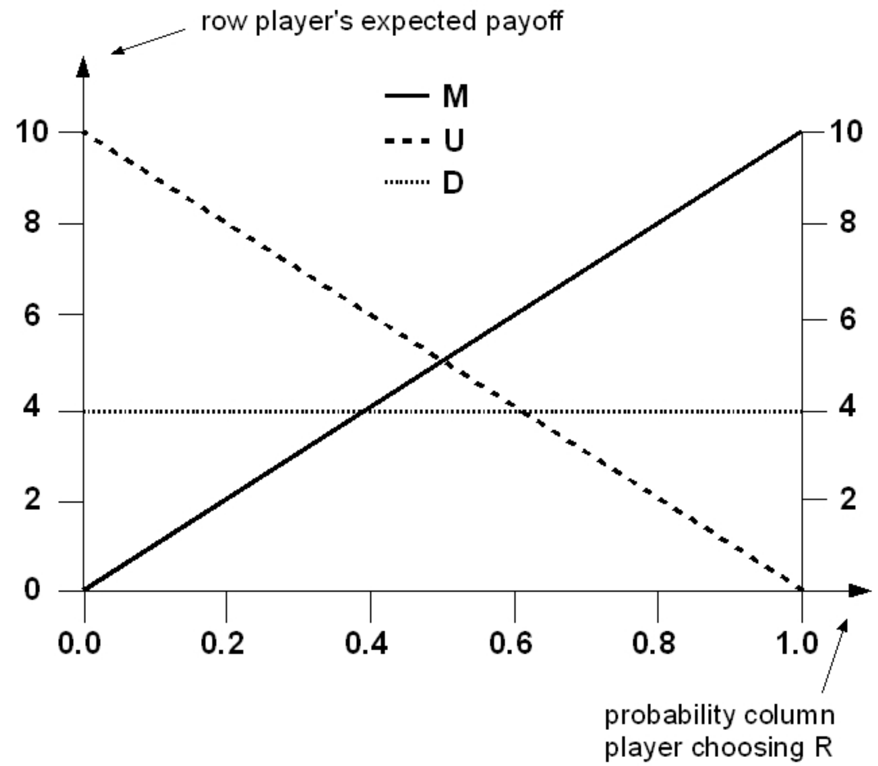
Note also: If **pure strategy** s_i **is strictly dominated** for player i, then **so is every mixed strategy** that **assigns** a strictly **positive probability to this strategy**.

Finally note: A mixed strategy can be **dominated by a pure** strategy **even if** all **strategies in its support** are **undominated** (see second example below)

Pure Strategies Dominated by Mixed Strategies

Example 11

s_1 \ s_2	**L**	**R**
U	10, ..	0, ..
M	0, ..	10, ..
D	4, ..	4, ..

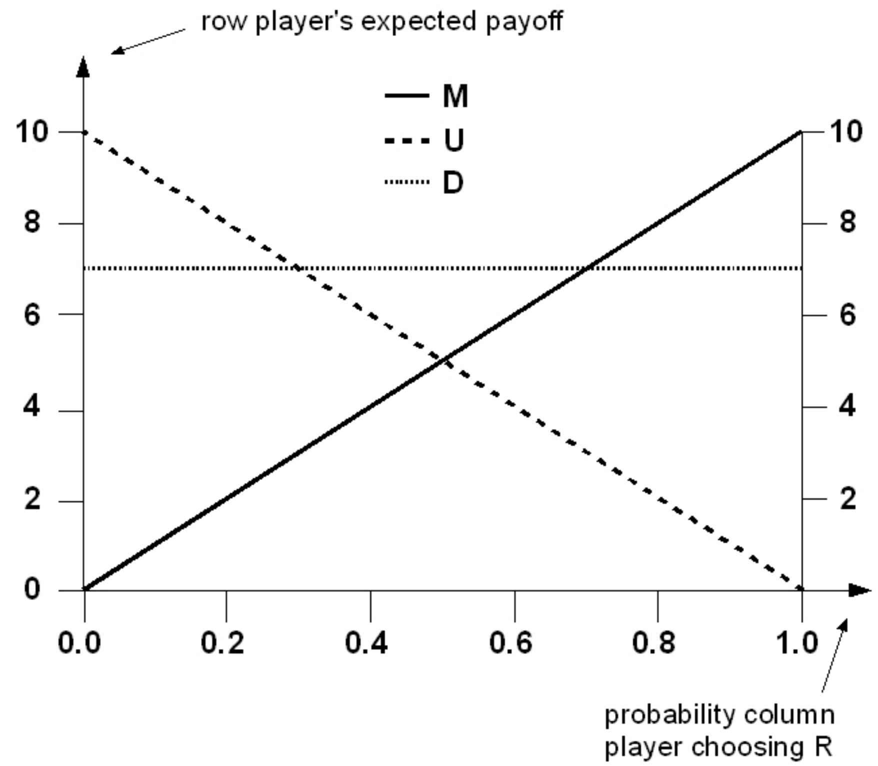


- row player (player 1) has three pure strategies: U, M, D
- none of these three pure strategies is strictly dominated by any of the others
- however, one of the three **pure strategies** is **dominated by** a **mixed** combination of the other two

Mixed Strategies Dominated by Pure Strategies

Example 12

s_1 \ s_2	L	R
U	10, ..	0, ..
M	0, ..	10, ..
D	7, ..	7, ..



- row player (player 1) has three pure strategies: U, M, D
- none of these three pure strategies is strictly dominated by any of the others
- however, there are **mixed strategies** which are strictly **dominated** by a **pure** strategy

Dominance: Concluding Remarks

The notion that (rational) players should play their **strictly dominant strategy** if they have one is compelling. However, strictly dominant strategies rarely exist.

The notion that (rational) players should not play a **strictly dominated strategy** is compelling, too. However, it is unusual for elimination of strictly dominated strategies to lead to an unique prediction for the game.

Iterated deletion of strictly dominated strategies sometimes is of some help but it has two drawbacks:

- each step requires a further assumption about what the players know about each others rationality.
- the process often produces a very imprecise prediction as well.

In the remainder of this lecture we study **equilibrium concepts.**

We show that (Nash) **equilibrium** is **stronger than** the concept of **iterated deletion** of strictly dominated strategies in the following sense: If a strategy profile is a Nash equilibrium (NE) then it survives iterated deletion, but there can be strategy profiles that survive iterated deletion but are not NE.

On the other hand, Nash **equilibrium** is **weaker than** the concept of **equilibrium in strictly dominant strategies**

Equilibrium Concepts

(Nash) Equilibrium: Strategy profile where no player has an incentive to deviate from his equilibrium strategy, given that all the other players stick to their equilibrium strategies.

	Static (Simultaneous) Games	**Dynamic (Sequential) Games**
Complete Information	Nash equilibrium	subgame perfect equilibrium
Incomplete Information	Bayesian equilibrium	perfect Bayesian equilibrium

We next turn to **Nash equilibrium (NE)**, the central solution concept in non-cooperative game theory. We begin with pure strategy NE in finite games (Lecture 3), proceed with mixed strategy NE in finite games (Lecture 4), and then turn to NE in infinite games (Lecture 5).

The other three solution concepts mentioned in the table are studied in later lectures, **subgame perfect equilibrium (SPE)** in Letures 6, 7 and 8, **Bayesian equilibrium (BE)** in Lectures 9 and 10, and **perfect Bayesian equilibrium (PBE)** in Lectures 11 and 12.